

***A Study of Students' Theoretical Thinking in a Technology-
Assisted Environment***

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Abstract

A study of Students' Theoretical Thinking in a Technology-Assisted Environment

Andreea Pruncut

This thesis analyzes students' theoretical thinking in a technology-assisted environment. The impact of such environment on students' thinking was studied on the sample of written solutions in a final examination in a Linear Algebra with Maple course. For analyzing students' solution a model of theoretical thinking proposed by Sierpinska in the context of her research on students' difficulties in linear algebra was used as a tool.

The motivation for undertaking this study was grounded in my belief that the use of technology in the teaching and learning of mathematics may turn students away from theoretical thinking towards a more action-oriented practical way of thinking. My study neither confirmed nor refuted this belief, but allowed me to refine my understanding of the nature of the influence technology may have on students' learning. It appears that whether the student's use of technology is effective or not depends also on the features of his/ her thinking (which may have been developed in previous study of mathematics). Students in my sample who approached the problems with a theoretical mind also knew how to make use of the computer in an effective way. On the other hand, students who demonstrated poor theoretical behavior could not control the software so as to arrive at a correct and complete solution. For these students, the computer acted as an obstacle

(called an “instrument-generated” obstacle here) that hindered the accomplishment of the task.

This study acknowledges the existence of certain phenomena that may arise in the process of working with the computer. Some of the findings reinforce or illustrate in the specific context of linear algebra various phenomena that mathematics educators had previously discovered in other mathematical subjects taught with other technology.

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Chapter 1: Introduction

In the last 20 years researchers have recognized linear algebra as a difficult subject to teach and to learn. They have acknowledged the fact that understanding linear algebra is cognitively demanding. In a theoretically framed linear algebra course, students need to think beyond the relations between individual matrices, vectors and operations and conceptualize them as algebraic structures such as vector spaces or normed vector spaces, and linear transformation amongst them. A particular source of difficulty resides in the variety of alternative languages that can be used to represent these objects, structures and transformations. Moreover, a good understanding of linear algebra requires theoretical thinking, which was described by Sierpiska (2000, 2002) as reflective, analytic, and systemic kind of thinking.

Today, the importance of technology in modern applications of linear algebra has been stressed by educators (LACSG, 1990). The possibilities of visualization and rapid computation that technology offers are believed to be effective in helping the students to construct mental processes necessary for learning linear algebra (Dubinsky, 1997). Computer Algebra Systems (CAS) have been already used in experimental classes in secondary schools in the hope that their judicious use will support investigative learning environments and thus improve students' understanding of important mathematical concepts and their applications (Artigue, 2005; Drijvers and Gravemeijer, 2005; Lagrange, 2005; Trouche, 2005) as well as their communication skills and attention (Bracewell et al., 1998).

The aim of this study is to examine students' thinking and understanding of linear algebra in a technology-enriched environment, using the model of theoretical thinking

developed by Sierpinska (2000) and Sierpinska et al. (2002). This model will enable us to get an insight into how students reason and engage in issues of mathematical exposition and communication. We find this theoretical model suitable for our purposes since learning linear algebra presupposes continuous reflection and critical attitude toward mathematical argument.

We think that it is important to describe and understand students' mathematical reasoning in a technology-enriched environment. Most of the teachers have not been specially trained to teach in a CAS environment; they learn and improve as they teach. This work could serve as additional information for instructors, who might be interested in studying students' mathematical behaviors and the possible specific difficulties they may encounter while working with CAS so as to prevent certain type of errors and guide students' instrumented work.

We have started this study assuming that a technology-assisted environment will support practical more than theoretical thinking. That is, we expected that students will undertake actions and focus on particular examples and objects and will spend less time reflecting and validating their work. The idea on which this supposition is based is that most people appreciate the computer for its computational efficiency: it performs long procedures faster. Therefore, in a computer-assisted course students may use the computer only as a calculator, and not necessarily for investigating conceptual ideas. Depending on the way it is taught, a technology-assisted course has the potential to favor this practical perspective, which is also increasingly advocated by partisans of "practitioner" as opposed to academic approaches to university education. In order to explore this hypothesis, we have analyzed several written solutions to a linear algebra

problem. Besides being interested in how students think, we wanted to know how students decide to use CAS, and if they write down all the justifications. We were interested also in seeing if the CAS environment creates any particular obstacles to the understanding of linear algebra. The samples of written work belong to students enrolled in an undergraduate linear algebra course where Maple was used by the teacher for writing and delivering lectures and by the students to do their homework assignments and write examinations.

To summarize, this study will try to answer to the following questions:

1. How do students in a computer-assisted linear algebra course think?
2. What features of theoretical thinking do students mobilize?
3. What kind of obstacles to the understanding of linear algebra do students encounter? Which of them are specific to the CAS environment?

Of course, these are broad questions that require more work than this study accomplishes, but even just initiating this enterprise could still be useful.

The thesis contains seven chapters: this introduction, presented as Chapter 1, followed by Chapter 2, in which we introduce the theoretical background of this study. We review the existent literature on teaching and learning linear algebra as well as the literature on technology in teaching mathematics, in Chapters 3 and 4. Our methodology of research and the results of the study are presented in Chapters 5 and 6. In the end, Chapter 7 contains our conclusions, the discussion of the results and some recommendations for teaching and future research.

Chapter 2: The Theoretical Background of the Study

This chapter introduces the theoretical foundations of this study. We begin by discussing how theoreticians and philosophers have acknowledged the concept of theoretical thinking in their works, followed by short presentations of the theories, which will be later mentioned in this thesis.

2.1 The Notion of Theoretical Thinking in Philosophy

The idea of a theoretical way of thinking is present throughout the literature from Greek thinkers up to our modern times. This way of thinking is often contrasted with practical thinking, which aims at undertaking an action and solves immediate problems. The art of keeping the balance between these two forms of thinking has always been an important concern for educators.

Aristotle's categorization of disciplines into theoretical, practical and productive was the ground for vast epistemological explorations. For him, theoretical sciences, such as theology, mathematics or the natural sciences, aim at truth and are pursued for their own sake. According to Aristotle, what we know scientifically is what we can derive, directly or indirectly, from first axiomatic principles. The highest form of human activity, the pursuit of truth through contemplation, is associated with theoretical sciences: thinking for its own sake, reflecting on one's already possessed knowledge. The role of educators should be, according to Aristotle, training people in the discipline of

contemplation. For the modern reader, contemplation may be equivalent to reflective thinking.

Practical sciences, originally associated with ethics and politics, are concerned with human action, and aim at cultivating knowledge. Practical reasoning has as starting point a particular question or situation. Aristotle saw *praxis* as guided by a moral disposition to act truly and rightly (*phronesis*). The form of reasoning associated with the practical sciences is *praxis* or informed and committed action.

The philosopher associated productive sciences with the thinking of a craftsman that has an idea of what he wants to make and uses some skills for bringing it into being. Besides technical inclination, the actions of a craftsman involve artistic creativity and the use of some theory that provides the basis for the action.

For Aristotle, *praxis* is the engagement of people with a situation as committed thinkers and actors. In essence, this definition seems to fit with what, today, we think practitioners should be: trained professionals who solve problems of practice (informed action) using the techniques they have learned at school.

Stepin (2005) does not speak of theoretical thinking, but of theoretical knowledge, which could be seen as a product of theoretical thinking. He describes theoretical knowledge as evolving from its primary forms (philosophical knowledge), into a developed form of knowledge “in which models of the object relations of reality are first created as if from above with respect to practice”(p. 373). Discussing theoretical knowledge as a result of the historical development of culture and civilization, Stepin clearly distinguishes two stages in the development of science: pre-science and “science in its own true meaning” (p. 20). In the stage of pre-science, the first ideal objects and

their relations were directly taken out of practice (through schematization of practice) and only later new ideal objects were formed, within the already created systems of knowledge. In science, initial ideal objects are not taken directly from practice; they are borrowed from previously developed systems of knowledge. Such models serve as hypotheses that later, after receiving justification, are transformed into theoretical schemes. In theoretical research, idealized objects are manipulated and new fields of objects are discovered before being assimilated into practice (p. 373). According to Stepin, for the transfer to scientific method of knowledge generation to happen, necessary socio-cultural premises were needed. Democracies of Ancient Greece created the norms of behavior and activity that made possible the development of philosophical cognition, where patterns of theoretical discourse appeared for the first time (p. 26). Stepin claims that the cognitive structure of a scientific discipline is determined by the levels of the theories and their relations to each other, the level of empirical research (facts and observations) and the links of these levels with the foundation of science. By philosophical foundations of science, Stepin means the ideals and norms that facilitate the integration of methods created by science into “the flow of cultural transmission” (p. 375).

2.2 The Anthropological Theory of Didactics

Chevallard (1999) claims that any purposeful human activity that is accomplished on a regular basis, and mathematical activity in particular, could be depicted using what he calls the praxeology model. The model identifies four elements of such activity. Two

of these belong to the level of *praxis* or *know-how*, namely the tasks that the activity aims at accomplishing and the techniques considered appropriate for solving them. The other two belong to the level of *logos* or the discourses that provide conceptual tools and procedures that justify the techniques (“technology”) and theories that justify the technology and provide the theoretical foundations for the activity (“theory”).

The pair technology –theory embeds the knowledge (logos and know-how) of the whole praxeology. Sometimes, a praxeology shows an uneven development of the two levels: the practical and the theoretical levels. Philosophical systems exist with almost non-existent practical level (there are no particular tasks to accomplish). There are also linear algebra courses that focus entirely on the development of the theory of vector spaces and linear transformations, without engaging students in solving the tasks and using the techniques that this theory is supposed to justify. This is not to say that vector space theory, which is a result of a long process of generalization and unification, is a theory without a task. The above-mentioned example illustrates a situation in teaching.

By contrast, there are linear algebra courses which focus on solving systems of linear equations using the Gauss-Jordan algorithm, determinants, Cramer’s rule, formulas for calculating areas and volumes and the computation of eigenvectors and eigenvalues, without spending much time if at all on the theoretical underpinnings of these techniques. The practice of using statistics in social sciences without questioning the models and techniques used is another example of working only at the practical level of concrete tasks and techniques.

As in case of Aristotle’s practical and productive sciences, for Chevallard, theory and practice are not opposite entities. Practice is or should be justified by theory, and the

rationale for developing a theory is or should be found in well-defined tasks and techniques. Reflection and action, ends and means should be in continual interaction. For educational practice, this means that the process of teaching a certain mathematical knowledge (including its practical and theoretical aspects) would do well to take into account six moments of study:

- (1) the first encounter with the knowledge,
- (2) exploration of the types of tasks related with this knowledge,
- (3) work on techniques, usually in interaction with
- (4) the construction of a conceptual frameworks and theory'
- (5) institutionalization and
- (6) evaluation (Chevallard, 2002).

These stages do not necessarily unfold in this order.

1. Theory of mathematics education, according to Chevallard, is the science of the diffusion of mathematical praxeologies, i.e. of making mathematical practices and their underlying theories known in a society. This science constructs, for this purpose, didactic praxeologies. As any other praxeology, a didactic praxeology consists of a certain system of tasks and techniques and has a technology-theory component. The praxeology of the teacher collects the whole body of tasks that concern the teacher as the main actor. The teacher is subject to the numerous institutional constraints, and has little freedom in choosing his or her tasks or techniques for accomplishing them. Thus any description of a teaching practice is inevitably a description of an institutional practice and not of an individual's practice.

2.3 The Theory of Distributed Cognition

It has been said that “some mathematics becomes more important because technology requires it, some mathematics becomes less important because technology replaces it, and some mathematics becomes possible because technology allows it” (Demana and Waits, 2000). No doubt, technology opens a new perspective on teaching and learning mathematics. In particular, teaching linear algebra in a technology assisted environment offers a variety of teaching-learning situations that could be further discussed and analyzed for the purpose of having a better insight into the role of computers and mathematical software as cognitive tools. The basic components of any practice – the tasks, the techniques of solving them, the know-how¹ and theories used to justify the techniques and their choice (Chevallard, 1999) - are changing when the practice of teaching linear algebra starts incorporating computer algebra systems.

In the computer lab, students have to cope with many situations, including those unexpected problems due to the constraints of human-machine interaction; therefore they often use their practical thinking at the expense of theoretical thinking. Still, students need to describe and explain these situations, and therefore bring into play theoretical thinking. In fact, we could say that in a laboratory-based course the thinking is often collective. There is an exchange of knowledge among individuals- not only among peers but also between the instructor and students. Many times students come up with technical knowledge, which is shared with the instructor who brings in turn conceptual, strategic, and theoretical knowledge.

¹ Chevallard uses the word “technologies” for know-how in his model; but in the present day English this word is often used to refer to computer technology, which would be confusing in this work.

Introducing technology in the classroom could have important consequences for the social aspects of learning. Researchers now claim that intelligence and cognitive activity are not exclusively the products of an individual mind, but rather distributed among the learners, the activity they are involved in, the tools they use, and the academic community and its achievements (Jonassen and Reeves, 1994). The distributed cognition is viewed as a process intertwined “with the specific features and qualities of the actual situation” (Keitel & Ruthven, 1993). Related ideas are expressed by Lave (1988) who speaks about the kind of thinking observed in everyday practice as being “distributed—stretched over, not divided among—mind, body, activity and culturally organized settings which include other actors.”

A number of researchers (Hutchins 1993; 1999; Lave, 1988; Pea 1993; Salomon, 1993) have contributed to the development of a theoretical framework of distributed cognition. The question they try to answer is how the social and the material are related to the cognitive activity of the humans, how the culture, history and context are linked with cognition (Hutchins, 1993, 2000). For example, Hutchins (1993) studied how navigating a ship is the result of people interacting with each other and the tools at their disposal. This collaborative environment is the ground on which the collective knowledge about how to navigate a ship is build.

The theory of distributed cognition (TDC) claims that resources are not peripheral to the mind, but play a fundamental role in the cognitive system. Resources change the nature and the function of the cognitive system while contributing essentially to the realization of competent actions. TDC is a cognitive theory, which states that cognitive processes take place not in the individual mind, but among many brains. Such processes

engage coordination between the internal cognitive structure and the environment. The main idea in TDC is that intelligence comes from interaction and the environment is an active resource for learning and reasoning. In other words, the environment becomes a cognitive tool in the sense of Jonassen & Reeves (1994) who define cognitive tools as “technologies, tangible and intangible, that enhance the cognitive power of human beings during thinking, problem solving and learning”. TDC is proposed as a suitable theoretical framework for understanding how computers and humans interact. Moreover, researchers view the theory of distributed cognition as a foundation for the future design of increasingly complex new technological tools (Hutchins, 1995; Norman, 1993; Saloman, 1993).

2.4 The Notion of Epistemological Obstacle in the Literature

In his book, *La formation de l'esprit scientifique*, Gaston Bachelard (1938) introduced for the first time the notion of epistemological obstacle in the philosophy of science. He was particularly interested in the historical development of physics, which he viewed as fraught with simplistic explanations of natural phenomena, that were nevertheless appealing to the mind and therefore all the more difficult to overcome by subsequent generations of physicists. He called the very general principles on which those explanations were built, “epistemological obstacles”. One example of such obstacle is the principle of deriving all physical knowledge from direct experience; another – the principle of explaining all phenomena of nature by one, general law such as “all bodies fall”. .

Although Bachelard explicitly wrote in his book that the development of mathematics was free from epistemological obstacles in his sense, later philosophers of

scientific knowledge started seeing similar processes also in the historical development of mathematics (e.g., Lakatos, 1976). Lakatos' work became very well known among mathematics educators in the 1970s and 1980s who have tried to adapt to their theories of learning mathematics and their explanations, in particular, of students' notorious difficulties with certain fundamental mathematical notions. The first to do this was Brousseau (1983) who defined an epistemological obstacle as knowledge that may be effective in some domain of problems or theoretical issues, but cannot be generalized without substantial modifications to a broader or different domain; if well entrenched in the mind or culture, this limited knowledge creates an obstacle to making the necessary changes of perspective or approach and makes it very difficult to solve the new problems. For example, Brousseau saw a parallel between children's difficulties with decimal numbers and the historical difficulties of the implementation of the decimal system and explained it by the existence and common use of heterogeneous units in measurement. Brousseau (1997) also extended the notion of obstacle to the development of new knowledge to obstacles other than epistemological and categorized obstacles into *ontogenetic* (having their roots into the limited cognitive capabilities of the subject), *didactic* (which can be explained by teaching approaches) and *epistemological*. As an educator, Brousseau was mostly interested in processes of overcoming obstacles, which, for him, require the same kind of work as constructing knowledge.

New kinds of obstacles arose in the context of mathematics teaching and learning in computer environments. Balacheff (1994) coined the term of "computational obstacle", referring to incompatibilities between concepts in the established mathematical theories and their representations in mathematical software. For example, a dynamic geometry

software may construct a triangle as an ordered triple of points, while order is not assumed in the Euclidean geometry the software claims to represent (Balacheff, 1994, p. 364).

In the context of CAS, Drijvers (2000) draws attention to the software-specific obstacles, which prevent the student from developing the utilization scheme that he or she has in mind. The researcher describes two types of obstacles: those related to the process of using the computer effectively, which he calls global obstacles (e.g. the black box effect and the limitations of CAS: the expertise of the user is needed to perform a task) and the local obstacles, linked to a particular piece of mathematics and its relation with CAS (e.g. the differences between the algebraic representations specific to CAS and those expected by the student; the multiple roles of letters in CAS, as well as the difference between numerical and algebraic solutions). According to Drijvers, obstacles appear when there is no equilibrium between the conceptual and technical aspects of working with CAS. This happens when students are familiar with the techniques of using CAS but have poor understanding of the mathematics involved, or the other way around, when the students are lacking in the necessary technical skills but have a grasp of the mathematical concepts. Both this author and Balacheff emphasize the role of these “computational” obstacles as opportunities for learning and discussion of the underlying mathematical ideas. Drijvers also recommends developing teaching strategies for confronting and overcoming the obstacles as they may lead to intense frustration and lack of success.

Chapter 3: Literature Review on the Teaching and Learning of Linear Algebra

It is generally recognized that linear algebra is a difficult subject to teach and a hard topic to learn for many students. Linear algebra has grown to be an active area of research in mathematics education only in the last 20 years, when researchers started to discuss about the roots of students difficulties, the teaching design for overcoming these difficulties, or the need of changing the curriculum. This review of literature will try to give an account of the most important and influential research papers and articles with focus on teaching and learning linear algebra. Most of them, proving an international research interest in this subject, were published together in (Dorier, 2000). However, some other papers published under the auspices of the International Commission on Mathematical Instruction (ICMI) and the Linear Algebra Study Group (LACSG) will be taken into account as well.

3.1 Sources of Students' Difficulties in Linear Algebra. Possible Explanations

One of the main sources of students' conceptual and cognitive difficulties that researchers (Robert and Robinet, 1996; Dorier, 1997, Dorier et al., 2000) bring into discussion is the unifying and generalizing character of linear algebra. Students in a first linear algebra course focused on the theory of vector spaces are overwhelmed by the formalism of the theory, the number of new definitions and the apparent lack of relation with their prior knowledge. For them, the benefits of the axiomatization of linear algebra

are not obvious. They do not have enough mathematical background to appreciate the use of the theory in different contexts. It takes an eye of an expert to take advantage of the simplification that a formal theory has to offer. The aforementioned conclusions have been formulated in the light of the epistemological analysis of the history of linear algebra (Dorier et al., 2000).

After 1930, the theory of vector spaces has been increasingly used as a unifying and universal language within a diversity of domains of mathematics: functional analysis, quadratic forms, geometry, etc. Although it did not help to solve new problems, mathematicians saw the theory of vector spaces as a model of simplicity and generality. Nevertheless, extensive work and a shift of perspective were needed for these “simple” ideas to come to light. Dorier and Sierpiska (2001) identify two stages in the process of construction of a unifying and generalizing concept: the first one is the “recognition of similarities between objects, tools and methods” (p. 257) followed by a reorganization of knowledge in order to make the concept explicit. Teaching such concepts to students in the first year of university could be very problematic, since the linear problems they encounter can be solved without using the axiomatic theory. Thus, students might find the learning of the formal theory of vector spaces unnecessary, or even meaningless.

Another source of students’ difficulties lies in the nature of linear algebra as a blend of languages (modes of description) and systems of representation. The coexistence of the abstract mode of the vector spaces theory, the algebraic mode of R^n and the geometric mode of two and three-dimensional spaces requires a cognitive flexibility that is difficult for students to achieve (Hillel, 2000).

Hillel discusses the problem of representation, namely the difficulties that students encounter when they have to deal with different basis-dependent representations of vectors and transformations. However, lecturers often ignore students' difficulties with representations. During one lecture period, they frequently move between notations and modes of description without warning students about the meaning of these shifts.

Moving from the vector space-theoretical representations to representations in R^n seems to be the most problematic; students often confuse a vector with its representation relative to a basis and have trouble with reading the values of linear transformation given by a matrix relative to a basis. Since these mistakes are persistent, the researcher draws attention to the existence of a conceptual obstacle rather than a difficulty related to the operationalization of a procedure. Hillel speaks about two types of epistemological obstacles. The first one stems from the tendency of thinking in a geometric context (arrows rather than vectors, axes rather than basis). The other comes from students' experience of working with n -tuples, which hinders the learning of the more general theory and the conceptualization of other mathematical objects - functions, matrices or polynomials - as vectors.

Lastly, researchers acknowledge the fact that understanding linear algebra is cognitively demanding. Terms like "trans-object level of thinking", "cognitive flexibility", "theoretical thinking", "structural mode of thinking" (Hillel, 2000; Dorier and Sierpinska, 2001) etc, have been used to describe the thinking required in learning linear algebra. Learning the general theory in linear algebra requires a trans-level of thinking. The term has been coined by Piaget & Garcia (1989) within their theory of mechanisms of scientific knowledge development, based on the notions of intra-, inter-

and trans-object levels of thinking. The inter-object level of thinking involves isolated forms of learning; an intra-object level is concerned with the relationships among these forms, whereas the trans-object level of thinking engages building and developing an entire conceptual structure. In the learning process, each level is continually revisited and revised. Novice students' performance often does not go beyond the inter-level of thinking. When a trans-level of thinking is required, students try to mechanically imitate the formal discourse they observed in the class or the textbook, without understanding its meaning (Dorier and Sierpinska, 2001; Hillel and Sierpinska, 1994; Hillel, 2000).

3.2 Researchers' Suggestions for Teaching Linear Algebra

There are two main orientations in the teaching of linear algebra at the undergraduate level: a theoretical approach focused on systematic development of vector space theory, where proofs play an important role, and a more practical, vectors-and-matrices approach, focused on computations and, sometimes, also numerical methods and applications.

The Linear Algebra Curriculum Study Group was founded in 1990 in United States. This group of mathematicians proposed a set of recommendations aiming at better addressing the interests of students. The group pointed to the potentially overwhelming effect of an increased level of abstraction in a beginning linear algebra course. They advised to carefully take into account students' mathematical background, and called for a shift in focus, more emphasis on problem solving, motivating applications, and "an awareness of the importance of technology in modern applications of linear algebra". A

first linear algebra course should proceed from concrete and practical examples to the development of general concepts and principles.

Harel (2000) proposed three principles for the teaching of linear algebra: the Concreteness Principle calls for a concrete context for students to build meaning and understanding; the Necessity Principle pleads for motivating the concepts in realistic context so that the students will feel intellectual curiosity to learn; lastly, the Generalizability Principle postulates designing instructional activities so as to facilitate generalization of concepts. Harel (2000) also advocates the teaching of basic linear algebra ideas since high school so that the first course in linear algebra could build on existing conceptual and procedural anchors. The researcher suggests that technology may be useful in meeting the conditions of the Concreteness Principle.

Chapter 4: Literature Review on Mathematics Teaching and Learning in the CAS-assisted Environment

Important research in the teaching and learning of mathematics assisted by computer algebra systems has been carried out in the past decade. The scope of this research can be gleaned from the recently published book “The Didactical Challenge of Symbolic Calculators – Turning a Computational Device into a Mathematical Instrument” (Guin et al. 2005). The book brings together works of French, Dutch and Australian teams of researchers who ask questions and propose answers on the subject of mathematics teaching and learning in symbolic computation environments, using mainly the symbolic hand-held calculators. But the results could be applied generally to CAS environments.

The purpose of this chapter is to outline several theoretical frameworks used in research on technology in mathematics education. The chapter is structured around issues regarding different types of uses of CAS in teaching and learning, different types of behaviors in CAS-assisted behaviors, and several didactic, cognitive and pedagogical matters.

4.1 Theoretical Frameworks Used in Research on CAS in Mathematics Education

Among the theoretical frameworks often used in research on technology in mathematics education we find the Anthropological Theory of Didactics (ATD), The Theory of Didactic Situations (TDS), Semiotics and The Theory of Instrumentation (TI).

While the first three have been elaborated in the context of the didactic of mathematics, the last one is a theory adapted from ergonomics. TI aims to study how students do mathematics in a technology-assisted environment where the mathematical knowledge is tightly connected with the instrumental knowledge.

4.1.1 Anthropological Theory of Didactics (ATD)

Several French researchers (Artigue, 2002; Lagrange, 2005; Trouche, 2003) have been using Chevallard's Anthropological Theory of Didactics as a theoretical framework to describe practices involved in the use of technology in mathematics education. In Chevallard's theory, mathematical thinking and the construction of its objects are outcomes of institutional practices or praxeologies. A praxeology is characterized by the types of *tasks* that the institution is called on to fulfill or undertakes to reach its goals; the *techniques* that the institution promotes and uses to solve each type of task, and the discourses that the institution uses to explain, justify and theoretically ground the techniques. Chevallard uses the word "technology"¹ to refer to the discourses justifying the techniques, and the word "theory" for discourses used to justify the technology. From the point of view of ATD, mathematics as practiced in mathematics departments is different from mathematics done in, say, a bank or an insurance company because they have different tasks to fulfill. Likewise, mathematics as taught and learned in a paper and pencil environment is based on a different praxeology than in a CAS environment: obviously, the techniques of solving mathematical tasks are different; perhaps less

¹ "Technology" is used here in a different sense than "electronic or digital products and systems considered as a group"(retrieved from the internet at [http:// www.dictionary.com](http://www.dictionary.com)). In this thesis, "technology" will be used in the latter sense, unless mentioned otherwise.

obviously, the mathematical tasks can be different as well. The pedagogical tasks may also be different in each case.

Researchers, who have been using ATD in their work, stress that the word “techniques” does not refer to mindless application of ready-made rules or formulas. Artigue (2002), for example, defines techniques as complex combinations of reasoning and routine work. Beside the pragmatic value given by its productivity, a technique also has epistemic value because, through the questions it raises, it contributes to understanding of the mathematical objects involved. Researchers have also been adapting ATD for the purposes of their research, stressing this or that aspect of the theory and operationalizing it. In particular, Lagrange (2005), who has been studying students’ and teachers’ uses of programmable calculators equipped with a CAS, reformulated Chevallard’s theory in terms of structural levels of mathematical activity:

- The level of tasks – general structures for problems – level of action
- The level of techniques – the organization of tasks – level of action
- The level of theory – theoretical discourse on the consistency and effectiveness of techniques – level of assertion

Lagrange used these levels to discuss both the support that technology affords mathematics teaching and learning and the specific difficulties that it brings about. One of the difficulties is the risk of loss of the epistemic value of techniques, since the traditional techniques are replaced by pushing buttons, or writing CAS commands. There is a need to endow the new techniques with epistemic value of their own. The new techniques cannot simply take over the role played in the conceptualization by the traditional techniques. The pragmatic and epistemic values of the paper and pencil

techniques have to be reconsidered, and new praxeologies must be developed. Mathematics educators have to agree on what are the necessary paper and pencil skills that teachers and students must have mastered even in a technology assisted environment. This is not an easy thing to do, Lagrange reasserts, since the mathematical culture and the paper and pencil techniques are tightly interconnected and the instrumented techniques are not believed to have an epistemic value. Yet, Lagrange argues that CAS techniques have the potential to assist mathematical conceptualization.

4.1.2 The Theory of Instrumentation (TI)

A theory of instrumentation proposed by Rabardel (1995; see also Rabardel & Bourmaud, 2003) in the frame of ergonomic theory has been successfully applied in research on technology in mathematics education by several researchers (e.g., Trouche 2003; 2005). The theory distinguishes between the tool, which is an *artifact*, i.e. a material object that an individual may use in performing a task, and an *instrument*, which is a combination of the tool with the psycho-motor and cognitive schemas developed by the individual for using the tool for certain purposes. The physical properties of the artifact constrain the schemas and the range of things that can be done with the tool. On the other hand, however, the potentialities of the tool broaden the scope of the individual's schemas of action and thought. The process of constructing an instrument from the artifact is called *instrumental genesis* and has two components:

Instrumentalization: the individual adapts the tool to his or her habits of work. This process develops in three stages: the stage of discovery and selection of those

aspects of the tool that are relevant for the work to be done; the stage of personalization of the tool, and the stage of transformation. Depending on the user, instrumentalization could make the artifact efficient or, by contrast, a burden.

Instrumentation: the process through which the tool structures the activity of the user. This process develops in two stages: the stage of “explosion”, where the individual discovers a multitude of techniques and strategies but doesn’t know how to use them effectively, and the stage of stabilization and purification, where the techniques and strategies become adapted to the constraints and potentialities of the tool.

Thus, the instrumental genesis involves the development of schemas in which technical and conceptual aspects interact. Researchers use the instrumentation theory to analyze the instrumented techniques and tasks in a technology-assisted environment (Lagrange, 2001, 2005) or to raise the awareness of the conceptual difficulties which become visible when students work with CAS (Drijvers and Gravemeijer, 2005). The theory emphasizes the role of the tool and how developing instrumental knowledge (knowledge about how to use the tool) may be tightly linked to the development of mathematical knowledge. The tool used to perform the tasks impacts on mathematical meanings. Thus, mathematical activity in a technology-assisted environment presupposes a good instrumentation of the tool.

4.1.3 Semiotics

Winsløw (2003) elaborates on a theory developed by Duval (1995, 2000) to discuss the potential and functions of CAS in undergraduate teaching. In Duval’s theory,

semiotic representations are defined as “productions made by use of signs belonging to a system of representation which has its own constraints of meaning and functioning”. Semiotic representations have an important role both in thinking and communication of mental representations. The theory distinguishes between *semiosis* – production of a semiotic representation – and *noesis*, which is a conceptual apprehension of the object of a sign. Semiosis leads to the development of “semiotic registers”, that is, systems of signs that allow the following three cognitive processes:

- *formation* of systems of signs;
- *processing* of the signs within the same system; and,
- *conversion* of signs from one system to another.

Duval stresses that, in the domain of mathematics, conversion is a non-trivial semiotic act, yet it is often taken as “natural” in teaching and not explicitly discussed or taught. This may explain students’ difficulties with, for example, moving between decimal and fractional representations of numbers, synthetic and analytic geometry, etc. Moreover, Duval develops also the notion of “discursivity” by defining four discursive functions: designating objects (referential), making statements about objects (apophantic), coherently developing statements (expansive), and reflecting on the value of statements (reflective). For a system of signs to be a language it must allow its users to perform these four discursive functions.

Winsløw (2003) suggests that some situations may allow the use of a CAS as an agent of conversion (in the sense of Duval, 1995). For instance, Maple’s three-dimensional capabilities facilitate the simultaneous visualization of infinity of solutions of an ODE - a whole family of examples is “encapsulated” into one single representation.

In his example, Maple is a mediator for coordination of and conversion between the algebraic and graphical registers. Winsløw (2003) notes that CAS may contribute to semiotic flexibility by facilitating processing and sometimes conversion (through plotting), but does not easily facilitate the coordination of registers, nor does it simplify the representation of objects. The fact that a CAS may act as an “automatic semiotic agent” could affect the mathematical discourse. Moreover, the special language of a CAS adds another dimension to the semiotic activity, which is difficult to handle by the inexperienced users. The author further discusses the potential of CAS for helping students operate at higher conceptual level (*the lever potential*) by leaving the lower level operations to the computer. He also mentions the potential benefits that students may gain when discussing the mathematical reality materialized on the computer screen (*the materialization potential*).

4.2 Types of Uses of CAS in Mathematics Teaching and Learning

With the software now available, students have the opportunity to explore a wider range of mathematical problems using multiple representations of objects while using both inductive and deductive approaches. Lagrange (2005) discusses two types of situations in which the machine is exploited differently:

- In one of them, students use CAS as a problem solving assistant (the machine performs the calculations) or as an aid to visualization and interpretation (the machine creates visual images or produces results that enable students to interpret and discuss them, such as graphically interpreting the zeros of a function).

- In the other type of situations, students use CAS to explore and discover a general structure by induction: even before knowing how to perform some techniques by hand, for example, techniques of differentiation and integration, students could use the machine to experiment with limits or derivatives in order to discover the algebraic rules behind them.

Situations of each type require adequate preparation, since, as predicted by Duval's theory of semiotic registers, students do not immediately switch between different representations (e.g., between analytic and tabular representations of functions; between approximate or exact values), nor do they readily think inductively. In order to start the process of instrumentation of a CAS, students have to know what to expect (e.g., whether exact symbolic values or approximate values), and also to understand how the machine simplifies an expression. The researcher suggests designing learning situations that challenge students to make predictions, and then compare their conjectures with the answers of the machine. Since CAS gives answers, but not insight into the process and helps to conjecture but hides the algebraic properties, he proposes a mixture of techniques: CAS for a local meaning of the solution and pattern discovery and paper and pencil techniques for insight into patterns and a general view. There is also the situation of CAS giving access to generalization: for example the case of optimization problems, where students could try to solve a numerical case and, constrained by the limitations of their calculator techniques, to search for a symbolic technique. Nevertheless, introducing parameters to generalize a numerical solution may yield difficulties linked to the various roles that a letter can play.

4.3 Types of Students' Behavior in a CAS-assisted Environment

In a teaching experiment involving 500 students over one year in a graphic/symbolic calculator environment, Trouche (2005) identified five types of student behavior:

- Theoretical behavior: use of mathematical references as systematic resource, with reasoning based on analogy and rigorous interpretation of facts; occasional use of calculators; medium command/control of the solution process
- Rational behavior: preference for paper and pencil, reasoning based on inference and proof, strong command/control of the solution process, reduced use of calculator;
- Automatistic behavior: use of cut-and-paste strategies from solutions saved in the machine's memory, trial and error procedures, poor command of the solution process, no strategies of validation, mixed sources;
- Calculator-restricted behavior: use of the calculator for investigations and as main resource, reasoning based on "accumulation of consistent machine results", weak command of the solution process;
- Resourceful behavior: use of all available resources, reasoning based on comparison of information, imaginative solution strategies, and medium command of the solution process.

Trouche emphasizes the limited nature of this typology, stating the difficulty of fitting a student's behavior within one single type. Other typologies of students' behavior in a technological environment have been proposed. For example, in relation with a

graphic calculator environment, Hershkowitz and Kieran (2001) distinguish between *mechanical-arithmetical* behaviors where students combine representations without thinking or *meaningful* behaviors where the choice and/or interpretation of representations are concept-based.

4.4 Didactic and Cognitive Issues

Drijvers and Gravemeijer (2005) define an instrumented technique as a set of rules and methods in a technical environment used to solve a certain type of problems. They are interested in the relation between instrumented techniques in the CAS environment and mathematical concepts. They conclude that CAS environment can foster awareness of steps in the solving process that are implicit in solving by hand (for example expressing one variable in terms of the others, or the idea that an equation is always to be solved with respect to an unknown) and moreover, the teacher could set these topics as a ground for discussions and turn some obstacles into opportunities for learning. In the end, it is not the CAS in itself that fosters the instrumental genesis, but the combination of CAS use, task design and educational decisions. Students can only understand the logic of a technical procedure from a conceptual background.

Lagrange (2005) discusses the potential of CAS techniques to have an epistemic role (i.e., a role in understanding and conceptualization), since they may help in understanding the structure and the equivalence of algebraic expressions. However, the student should learn how to use CAS functions effectively, they should learn to anticipate the outputs and decide if two expressions are equivalent or not. The teachers

(mathematics educators) would have to consider the impact of technology on existing techniques, and design new techniques as bridges between tasks and theories. This is difficult to achieve, since institutional obstacles may arise as the values of the school are related to the paper and pencil techniques.

Artigue (2005) notes that the modes of reasoning behind the algebraic computations are usually ignored, as they are limited to the execution of algorithms. In a paper and pencil environment “the intelligence of algebraic manipulations” is not likely to develop because algebraic manipulations are reduced to a small number of routines. The computer makes it possible to work with complex expressions. However, the inputs and the outputs are different from what students in a traditional environment are used to. Students are challenged to understand how the results returned by the computer relate to the paper and pencil techniques. The commands have to be suitably understood and adapted to fit the tasks while an object has to be identified as having different representations.

Examining the experimental works of other researchers, Artigue (2005) points to three didactical strategies that were used in the experiments:

- *The surprise lever*: exploiting the effect of unexpected results to undermine the flaws in students’ thinking, provoke questions and motivate student’ work.
- *The multiplicity lever*: using the potential of technology to produce numerous results in a short time so as to motivate and promote the search for regularities, invariants and their understanding.
- *The dynamic lever*: using the graphical capabilities of the machine to promote a dynamical way of approaching the mathematical concepts.

Artigue claims that instrumental genesis and mathematical knowledge could both develop together as tightly interconnected parts of the same “integration process”. Some issues difficult to manage in a traditional environment, like the algebraic sense or the relationship between exact and approximate computations are easier to address in a technology-assisted environment. The opportunities to visually motivate the introduction of new concepts, as well as the potential for generalization (for example, in the case of optimization problems) are other CAS benefits mentioned by the researcher.

Winsløw (2003) points to some pitfalls to be considered and controlled in teaching:

- *The Jourdain effect*¹ – when students do not have an active participation in the “higher level” (conceptual) discourse, but they only formally perform semiotic actions with CAS. The teacher tells them what to do and what to expect, and interprets their results.
- *The animator effect* – when the teacher is just an animator while students are engaged in mathematically irrelevant activities
- *The empirical bias* (the particularity issue) – using CAS may favor the shift of attention to particular examples, and the development of the inductive forms of thinking at the expense of the deductive ones. The author gives an example from linear algebra, where the use of Maple was intended to illustrate the meaning and the use of the eigenvectors, but some students used it to recognize the pattern on the basis of several particular examples.

¹ For the concept of “Jourdain effect” see Brousseau, G. (1997). *Didactical Situations in Mathematics*. Dordrecht: Kluwer Academic Publishers.

- *The black box effect* – the CAS user’s lack of access to the intermediate steps of a solution (only the inputs and outputs are visible, which may favor the idea that one does not need to understand the algebraic steps involved). A strategy like trying more inputs to compare the outputs could not necessarily stimulate the lever potential, but induce phenomena like “localized determination” (Trouche 2005), in which students try repeatedly the same strategy even when no relevant results are produced.

As for teaching design in the CAS-assisted environment, Winsl w recommends in-class discussion about the intermediate steps of processing and their role in mathematical discourse, as well as introducing problems that require the coordination of algebraic and graphical registers and conversion from one to the other. This will encourage a more reflective and elaborate mathematical discourse.

4.5 Pedagogical and Institutional Issues

Integration of technology in the classroom depends on the coordination of factors such as learning goals, available instructional tasks, didactic contract (Brousseau, 1997), socio-mathematical norms (Yackel & Cobb, 1996), students’ prior algebraic skills as well as teacher’s guidance. Still, the complexity of mathematical software is often at odds with the conceptual clarity of traditional mathematics teaching, and therefore to balance this situation, teaching basic ideas of data processing is often suggested. For the integration of CAS, researchers also recommend introducing realistic problem situations from the beginning, to make the transition from informal meanings and strategies to formal methods more meaningful (Drijvers and Gravemeijer , 2005).

Winsløw (2003) argues that the use of CAS may generate “conflicting intentions” among teachers and students. Some students may be extrinsically motivated to use CAS by the desire to minimize the time spent on a course and still have satisfactory results. On the other hand, Trouche (2005) speaks about the risk of students being confused and frustrated due to learning in two environments: traditional and computerized, especially when computers are not available outside the lab.

Guin, Ruthven and Trouche (2005) recommend introducing new explicit teaching objectives to promote the development of an algorithmic spirit and allow experimental approaches and new forms of working (groups of students involved in interdisciplinary projects on a given theme). It is also very important to allow time for the didactical management of these objectives, as well as to design appropriate assessments that will involve more experimental approaches and open-ended problems and the use of instrumented techniques. This is to evaluate students’ capacity to make conjectures, to interpret the results of the machine, to evaluate the relevance of a tool and to coordinate different functions of CAS in order to validate an answer.

Artigue (2005) notes that the experiments carried out in a symbolic calculator environment did not acquire a mathematical status, and their epistemic value remained limited. This is partly due to the absence of a theoretical discourse that has to be developed to integrate both mathematical and technical knowledge and to support the institutionalization of the instrumented techniques. Besides the official discourse, there is also the need of trained teachers. Understanding the potential of symbolic tools for the learning and teaching mathematics requires reflection on the potential epistemic value of instrumented techniques. It is also necessary to take into account the connection and the

complementarity between paper and pencil techniques and instrumented techniques as well as the institutional negotiation of certain mathematical requirements of the instrumented work. Each technique has its own epistemic value and pragmatic value and the institutional status of the techniques depends on these values. If the technique is not credited as having epistemic value, it might not be seen as legitimate. While the pragmatic value of an instrumented technique is clear, the possible epistemic value is not always evident. The author recommends reflection and reorganization of tasks to make visible the possible epistemic value of the instrumented techniques.

Chapter 5: Methodology of the Research Study

In this section we will be introducing the participants and the sources of data, followed by a discussion of our procedure of analyzing data, which relies extensively on the model of theoretical thinking developed by Sierpinska et al. (2002). Using the usual criteria for assessing a qualitative study, we will also try to explain how our study fits (or not) into these criteria.

5.1 A Qualitative-interpretive Study

This study has features of both qualitative and quantitative research paradigms, but the quantitative aspects are too modest in scope to allow generalization or predictability of our findings. This is an exploratory study, which examines a few students' mathematical thinking in a CAS-assisted environment, looking particularly at the theoretical versus practical aspects of this thinking. Thus, it is closer to the qualitative-interpretive research paradigm, where, in a process of disciplined inquiry, the researcher seeks for a better understanding and extrapolation to similar situations and the intent is discovery rather than confirmation of some hypotheses.

Our search was structured by the model of theoretical thinking developed by Sierpinska et al. (2002). These authors' definition of theoretical thinking, which has an axiomatic character, was inspired by both the existing literature and empirical research of students' mathematical thinking. Delimiting phenomena into categories and making a list of behaviors to be observed are methods of research usually used in quantitative studies. In this respect, our study could be labeled as having quantitative aspects.

Terms such as reliability (the research results are replicable and stable in time) and validity (the methods of measurements are accurate and appropriate for the case under study) are specific to quantitative research. Within the qualitative paradigm, the researchers prefer to use categories such as dependability rather than reliability, and trustworthiness rather than validity. The consistency of data in a qualitative research is confirmed through close examination of raw data (Campbell, 1996). The researchers often speak of triangulation, an idea that comes also from the qualitative-interpretive paradigm. This is a further test that search for convergence of different methods and sources of data.

Mathematics education researchers base their findings on conceptual analyses of students' work. What students say or do is potential data for researchers' inferences, which are then used to confirm or reject certain hypotheses. There is (or should be) an effort to make explicit the criteria on which such decisions are being made (Goldin, 2000). Usually, the criteria depend on the theoretical framework within which the research questions have been formulated. Thus, this theoretical framework must be clear in a report of the research. Moreover, research methods and procedures are (or should be) described in sufficient detail to permit further use or reproduction of the procedures of data collection and their analysis (reproducibility). The description of the conditions, observations and inferences should be sufficiently precise to allow comparison with other studies (comparability).

Does our study satisfy the trustworthiness criterion? We try to satisfy this criterion by grounding our findings in the data. In order to make our study credible we provide a detailed analysis of each student's written solution. We hope to offer enough

information that can be used by the reader to judge whether the findings could be extrapolated to new situations. We also hope that the nature of this analysis (which has fixed categories to code) helped to keep at least some of our biases and perspectives at bay.

5.2 The Theoretical Framework of Our Research

The distinction between theoretical and practical thinking

The model of theoretical thinking compares and contrasts theoretical thinking (TT) and practical thinking (PT). This model states that TT *is reflective*: it intends to understand experience and reflect on the possible outcomes of an action (“what will happen if we do this or that?”), whereas practical thinking aims at undertaking an action (“what shall we do next?”).

Moreover, TT is *systemic*; it refers to systems of concepts and the meanings of the concepts and its focus is on developing relations between concepts within a system of concepts. On the other hand, practical thinking is about particular objects and the meaning of action and its focus is on particular examples and personal experience.

The main concern of TT is the epistemological validity, with the conceptual coherence and the internal consistency of the system and what is hypothetically possible. PT is concerned with the factual validity, namely what is plausible and realistic. The levels on which TT operates are the level of thinking about concepts and the meta-level of thinking about thinking, whereas the PT operates on the level of actions. In a

conceptual system, the theoretical thinker will be aware of the conditional character of the truth and will consider all possible cases, even when these seem not likely to occur.

Lastly, theoretical thinking is *analytical*. Sierpiska et. al. define analytical thinking as thinking that takes an analytical approach to signs. That is, in theoretical thinking the relations between the object and the sign that symbolically represent it, is mediated by a certain language, and this language is in itself an object of analysis. In mathematics, a theoretical thinker will be sensitive to the meaning, the role and the notation of the existential and universal quantifiers. Sensitivity to mathematical language entails also an understanding of the difference between a statement and its converse as well as recognizing a conditional statement even when it is not explicitly formulated as an “if ... then ...” statement. Moreover, the sensitivity to mathematical language encompasses differentiating between definitions, axioms, theorems, proofs and examples. Sierpiska et al. (2002) conclude that the analytic thinking has two aspects: linguistic sensitivity (to the mathematical syntax and terminology) and meta-linguistic sensitivity (to the symbolic distance between an object and the sign used to refer to it, and to the logic and structure of mathematical language).

The model of theoretical thinking has been created within the context of linear algebra, where there are many axiomatic definitions. The authors emphasize that the understanding of axiomatic definitions is very important in linear algebra since this would be necessary both to verify axiomatic properties of a given mathematical object (for example a vector space, or an inner product) as well as to construct new examples of objects having certain axiomatic properties. The axiomatic algebraic structures introduced in linear algebra, and generally, the unifying and generalizing character of the

modern theory of linear algebra require a certain level of formalism and the use of various languages and registers, which in many aspects, involve reflective, systemic, and analytic thinking.

Sierpinska et al. (2002) claim that, for understanding linear algebra, a student should be theoretically inclined; the learner should have a relational understanding of concepts, which can only emerge from knowledge and understanding of definitions and theorems. This is what the authors call a “systemic approach to meanings”. Moreover, the learner should engage in hypothetical thinking by asking “what if (we modify the assumptions)” type of questions. Since linear algebra is a combination of geometric, algebraic and abstract languages, the learner should be able not only to use these languages, but also to reflect on their structure.

The table bellow summarizes the features of theoretical thinking as described in the Sierpinska et al. model.

TABLE 1. A MODEL OF THEORETICAL THINKING IN MATHEMATICS

TT is reflective	TT aims at producing further knowledge; has a disposition toward inquiry; is creative because generates new ideas, but also critical because it continuously evaluates these ideas.
TT is systemic	TT is based on systems of concepts, where the meaning of a concept is understood in relation with other concepts.
- systemic-definitional	TT founds meaning on definitions; both the nature and the uses of definition are understood. In particular, the student understands that an axiomatic definition postulates a kind of mathematical objects rather than describes objects that already exist.
- systemic- proving	To establish the validity of a statement, TT engages in proving, using an explicit system of concepts,.
- systemic-hypothetical	TT analyzes every logically conceivable case and is aware of the conditional character of the statements.
TT is analytic	TT takes an analytic approach to signs and treats mathematical notation and terminology as elements of a specialized language, which is itself an object of evaluation and analysis.
- analytic linguistic sensitivity	TT is sensitive to specialized terminology and formal symbolic notations.
- analytic meta-linguistic sensitivity	TT distinguishes between the mathematical object and the sign, which symbolically represents it. Theoretical thinker is sensitive to the structure and logic of mathematical language.

A finer distinction: analytic-arithmetic and analytic- structural modes of thinking

Even before creating the model of TT, Sierpiska (2000) introduced a finer distinction, which has been developed especially for characterizing students' thinking in linear algebra: the analytic-arithmetic (A-A) and the analytic-structural (A-S) modes of thinking. Both these modes belong to analytical – and therefore theoretical – thinking. In some cases, the theoretical thinking model may prove to be too coarse-grained or too general to describe students' thinking in linear algebra and therefore a more detailed categorization is needed.

The distinction describes A-A thinking as aiming at the accuracy of calculations and their simplification, whereas the goal of A-S thinking is to enrich one's knowledge about concepts. In A-A mode of thinking one uses formulas to define and describe mathematical objects, while in A-S thinking a set of properties is used to describe an object. For example, to check whether two matrices are inverses of each other in A-A mode, one would attempt to calculate the inverse of one matrix and see if a solution exists and is equal to the other matrix. In A-S thinking, one would refer to the definition of invertible matrix, multiply the two matrices and check if the identity matrix obtains.

In the history of linear algebra, the A-A thinking preceded A-S thinking. The results of the first one, which are the computational techniques and methods, were structuralized in A-S thinking in more simplified, elegant and general definitions and proofs. This unifying and structuralized way of thinking has given birth to the modern theory of linear algebra.

5.3 Sources of Data

Our research is a modest study in terms of the amount of data and the costs of their collection. The data consisted of the final examination solutions of students enrolled in the second term of an undergraduate *Linear Algebra with Maple*¹ course. Maple is a CAS that includes a programming language, a large array of mathematical techniques, powerful dynamic graphing capabilities and can also be used as a word processor. This part of the course contained such topics as inner product spaces, normed vector spaces, Gram-Schmidt orthogonalization in inner product spaces; quadratic forms, QR-decomposition, orthogonal complements, orthogonal matrices and operators, symmetric matrices and self-adjoint linear operators, the Spectral Theorem, the method of Least Squares, singular value decomposition, pseudoinverse of a matrix. Students' mathematical thinking and approaches to solving the final examination questions were certainly dependent on the content of the course and the instructor's didactical choices. Therefore one has to be very cautious in generalizing the observations to students' mathematical behavior in any linear algebra with Maple course.

The final examination paper contained two sets of twelve questions each, with different weights in terms of marks. The students were to choose enough questions to make up 100 marks. The two types of questions were:

- T1-T12, where the teacher made it clear that the use of Maple “is not very useful or even completely useless”.

¹ “Maple” is a CAS developed in Waterloo, Ontario, Canada. Information about the software can be found at <http://www.maplesoft.com/index.aspx>. At the time of the course, version 10 of the software was used, mainly the “linalg” and “LinearAlgebra” packages.

- M1-M12, which the teacher introduced as problems where the use of Maple “can be useful or even necessary”.

After analyzing the problems, we chose to discuss here in more detail students’ solutions to the problem M2. Here is the text of the problem.

Let A be a 3×1 matrix, and let $B = AA^T$.

(a) Prove that B is a symmetric matrix.

(b) Prove that $f: R^3 \times R^3 \rightarrow R$, defined by $f(v, w) = v^T B w$ is not an inner product.

We chose one of the “M” problems because we were interested in students’ mathematical behavior within a CAS environment. Among these, M2 appeared to discriminate well between theoretical and practical thinking in solving linear algebra problems within a CAS environment, which is what we were interested in our research. Eight students (out of 22) decided to solve this problem.

Students’ solutions have been analyzed using Sierpiska et al.’s (2002) framework for identifying features of theoretical thinking in mathematics. This framework was used to first make some theoretical predictions about possible correct solutions of the problem and the theoretical thinking features that this problem could, in principle, mobilize. This theoretical analysis then served to describe students’ solutions. We explain below how this was done.

A theoretical analysis of the problem M2

Part (a) of the problem could be approached in two ways:

- “By inspection”, using the visual perception of the symmetry of the

matrix: calculating $AA^T = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$, where $A = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is any real 3×1

matrix, and observing that the matrix is obviously symmetric.

The proof lies in the result itself, and students only need to know how a symmetric matrix looks like. Students who would prefer this method might not even think of the structural definition of the symmetric matrix (the matrix A is said to be symmetric if $A = A^T$). This solution is done from the analytic-arithmetic perspective (in the sense of Sierpinska, 2000¹), and this is the only aspect of theoretical thinking involved in this solution. If a student chose to take a concrete 3×1 matrix A with numerical entries, and check if $B = AA^T$ is symmetric, then there would be no theoretical thinking involved; however, this kind of solution would not be awarded full marks.

- By proving that B satisfies the definitional property of symmetric matrices; this proof requires the use of the properties of the transpose operation on matrices: $B^T = (AA^T)^T = (A^T)^T A^T = AA^T = B$, hence B is symmetric.

This solution involves theoretical thinking, namely the following features: *systemic-definitional, systemic-proving, analytic-linguistic and meta-linguistic sensitivity*.

¹ Sierpinska, A. (2000). On some aspects of students' thinking in linear algebra. In J-L. Dorier (Ed). On the teaching of linear algebra (pp. 209-246). Dordrecht: Kluwer Academic Publishers.

Part (b) requires students to engage in proving or, alternatively, in axiomatic reasoning and proving to demonstrate that a certain type of function can never be an inner product. Two solutions paths will be presented bellow.

- Possible solution 1: Proving by checking the axioms of the inner product, using analytic-arithmetic thinking

Students could check whether or not $f: R^3 \times R^3 \rightarrow R$, defined by $f(v, w) = v^T B w$ has all the defining properties of the inner product and find that the positive definite property is not

satisfied: $f(v, v) = v^T B v = (ax + by + cz)^2$ is not necessarily positive if $v \neq \mathbf{0}$, $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

For instance, $f(v, v) = 0$ for $v = \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}$. If a and b are not simultaneously 0, this is a

sufficient proof that f is not positive definite. If $a=b=0$ and $c = 0$, then it is obvious that f is not positive definite because $f(v, v)=0$ for any vector v . If $a=b=0$ and $c \neq 0$, then one

can take $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $f(v, v)=0$.

This method would require engaging with the analytic – arithmetic mode of thinking, where to prove that X is Y , it is enough to show by various calculations that X has the same properties as Y .

Noticing that $f(v, v) = v^T B v = (ax + by + cz)^2$ makes the solution of the problem faster, but it requires either enough algebraic practice to see that $a^2x^2 + b^2y^2 + c^2z^2 + 2abxy + 2acxz + 2bcyz = (ax + by + cz)^2$ or analytic-structural thinking to see that

$v^T B v = v^T (A A^T) v = (A^T v)^T (A^T v) = (A^T v)^2$ since $(A^T v)$ is a number. Today's students usually lack the algebraic experience to notice the former relationship and are not highly likely to engage with the analytic-structural mode to notice the latter one. But it is not necessary to reduce the problem to the equation $ax+by+cz=0$ to find suitable non-zero vectors v for which $f(v,v) = 0$. Students can still figure out such vectors and obtain a correct solution, provided they understand that they have to prove the existence of such vectors v for any values of the variables a , b and c in the expression of A . Stating, for example, that $f(v,v) = 0$ for $a = b = c = 1$ and $x = 1$, $y = -1$ and $z = 0$ is not a correct solution. But this understanding of the status of the variables a , b and c as *arbitrary* in the problem requires certain features of theoretical thinking (systemic-hypothetical, systemic-proving and analytic metalinguistic sensitivity) and therefore even this solution cannot be obtained without some theoretical thinking.

- Possible solution 2: Proving by using a powerful theorem linking positive definite property with eigenvalues of a matrix.

Students could resort to the theory and use the theorem which says that a real symmetric matrix is positive definite if and only if its eigenvalues are positive, together with knowing that the non positive definite matrix $B = A A^T$ makes $f(v, v) = v^T B v$ not necessarily positive when v is a non zero vector and therefore $f: R^3 \times R^3 \rightarrow R$, defined by $f(v, w) = v^T B w$ is not an inner product. This method calls for an analytic-structural type of thinking (in the sense of Sierpinska, 2000), where the focus moves from calculating to prove a property to proving that an object has the property.

In terms of calculations, the approach that uses eigenvalues is more economical, although involves more structural knowledge. Even without Maple as a calculator, to find

that zero is an eigenvalue of B , it suffices to notice that B is singular since its rows are linearly dependent. Therefore, on account of the theory (“a real square matrix B is invertible iff 0 is not an eigenvalue of B ”), zero has to be one of the eigenvalues.

Whichever approach students take to solving part (b), they have to engage in theoretical thinking. A priori, 5 out of the 6 features of theoretical thinking could be revealed in one or the other of the above-mentioned approaches.

- Systemic-definitional (S-D): the student understands and uses the definitions of the symmetric matrix, positive definite matrix, inner product as well as the relation between positive definite matrices and the axioms of the inner product
- Systemic-proving (S-P): the student engages in proving and axiomatic reasoning to explain why the given function cannot be an inner product
- Systemic- hypothetical (S-H): the student is aware of the conditional (if-then) character of the statements to prove; in particular, he or she is aware that the statements to prove are true because of the very particular structure of the matrix A (a 3×1 matrix) and of the matrix $B = AA^T$.
- Analytic-linguistic sensitivity (A-L): the student uses coherent and correct terminology and maintains control over formal notations.
- Analytic-metalinguistic sensitivity (A-ML): the student is sensitive to logical aspects of mathematical statements; in particular, the student interprets the statement in part (b) as if he or she noticed the implicit general quantifier over the 3×1 matrix A and assumes that A is an arbitrary such matrix.

We decided to structure the data and the analysis so as to examine the occurrences of student's theoretical behavior (TB), student's non-theoretical behavior (nTB), but also the violation of the features of TB. We will grant the score 1 each time an expected theoretical behavior does occur and the score 0 otherwise. Similarly, when a non-theoretical behavior or a violation of TB happens, the score is 1, and 0 otherwise. We will justify our decisions in the "comments" column.

For example, if the student correctly uses the definitions of the symmetric matrix, positive definite matrix, inner product as well as the relation between positive definite matrices and the axioms of the inner product, we will code his or her theoretical behavior with respect to S-D feature with the score of 1. If his solution reveals an incorrect conception, or an incorrect use of at least one definition involved, or the theoretical feature is completely absent, the score would be 0. If the student's written solution shows evidence of a behavior that is in clear conflict with a certain feature of TT, we will assign for this violation of a feature of TT the score -1. If the theoretical behavior is not present at all, but also not violated, we will still have to give it the score of 0.

The results of this analysis will be presented in a table (see Table 2).

TABLE 2. FORMAT OF THE PRESENTATION OF THE RESULTS OF ANALYSIS OF STUDENTS' SOLUTIONS

Student	Student's theoretical behavior	Student's non-theoretical behavior	Student's violations of theoretical behavior	Comments
[Student's nickname]	[Description of TB in terms of scores on the various aspects of expected theoretical behavior : S-D, S-P, S-H, A-L, A-ML]	Description of SnTB in terms of scores on various features of non-theoretical behavior	Description of violations in terms of scores on the violation of various expected aspects of TT	Justifications

Chapter 6: Results of the Study

In this chapter we will present and analyze the solutions of each of the eight students who chose to solve the examination question “M2” described and analyzed in the previous chapter. For us, each solution has been a case in itself and therefore we tried to pay attention to every single detail of the written work. Although we searched to find evidence for certain fixed categories defined within the theoretical thinking model, we have been always aware that we might come across interesting aspects of students’ mathematical behavior, not captured in the model.

The Case of Student 1(Sabet)

Sabet chose to solve three T-questions and six M-questions: thus only 33% of the problems she chose to solve were of the more theoretical kind. We could say that she favored the Maple-assisted questions. She also wrote all her solutions in a Maple file, using the software both to write and to perform arithmetic and algebraic calculations for her; this counted, for us, as an aspect of her non-theoretical behavior.

In part (a) she did not take an arbitrary matrix A but chose to generate a random 3×1 matrix, with entries restricted to the interval $[1,4]$, using Maple. She obtained the one-column matrix with entries $\langle 3, 2, 2 \rangle$. Thus, for her, randomness was a substitute for generality: a kind of “statistical” generality. We consider this to be a symptom of a violation of the systemic-hypothetical feature of TT; we assign the score of -1 to S-H in the rubric “violation of theoretical behavior”.

She then used Maple to calculate the eigenvalues of the matrix $B=AA^T$, and obtained one double eigenvalue equal to 0 and another eigenvalue equal to 17. Her conclusion was: “ B is symmetric because the eigenvalues are all positive”. Being theoretically incorrect – having positive eigenvalues is not a sufficient condition for a matrix to be symmetric – this response reveals a lack of systemic-definitional thinking (S-D=-1 in the rubric of violation of theoretical behavior). Moreover, claiming that 0 is a positive number shows a lack of linguistic sensitivity, whence the score of -1 on the feature A-L in the violation of theoretical behavior rubric. Were the eigenvalues of the matrix B all positive, the function $f(v,w)=v^T B w$ would have been an inner product, contradicting the implicit claim in part (b) that this is not the case. Not having noticed this contradiction is a symptom of lack of sensitivity to logic and implies a score of -1 on the feature A-ML. The student is obviously unaware of this implication, and works under a rather different set of beliefs about the relationship between eigenvalues and the positive definite property.

In part (b) – proving that $f(v,w)=v^T AA^T w$ is not an inner product – the student took two arbitrary vectors v and w and defined the function f on pairs of vectors in Maple as follows, using correct syntax:

$$v:=<v1,v2,v3>:$$

$$w:=<w1,w2,w3>:$$

$$f:=(v,w)-> \text{Transpose}(v).B.w;$$

She then appeared to proceed to verify if f satisfies the axioms of inner product; this behavior was coded as systemic-proving and granted the score of 1 on this feature (S-P=1 in the TB rubric). She first verified if f is symmetric, having Maple calculate $f(v,w)$, then

$f(w, v)$ and then $f(v, w) - f(w, v)$. The output was long and messy, so the student asked Maple to “simplify” the expression and the output she obtained was “0”. This apparently satisfied her because she went on to calculate $f(v, v)$, i.e. – apparently – to verify the positive definite property. As a result, a long and complicated expression appeared:

$$v1 (9 v1 + 6 v2 + 6 v3) + v2 (6 v1 + 4 v2 + 4 v3) + v3 (6 v1 + 4 v2 + 4 v3)$$

which she tried, again, to “simplify”. This resulted in only a slightly simplified expression

$$9 v1^2 + 12 v1 v2 + 12 v1 v3 + 4 v2^2 + 8 v2 v3 + 4 v3^2 \quad (*)$$

Thus, Maple did not produce an explicit square of a trinomial, as could be theoretically predicted. She called this expression “eq1” (without equating it to 0, so, technically, there was no equation) and then put the command “solve({eq1},{v1,v2,v3});”. Maple treated the command as syntactically correct, with the default equation being [expression]=0.

The output was:

$$\left\{ v1 = -\frac{2}{3} v2 - \frac{2}{3} v3, v2 = v2, v3 = v3 \right\}$$

$$\left\{ v1 = -\frac{2}{3} v2 - \frac{2}{3} v3, v2 = v2, v3 = v3 \right\}$$

This duplication may appear surprising, but it is a logical consequence of the fact that the expression (*) is the square of the linear expression $3v1 + 2v2 + 2v3$. For a Maple-literate person this double output would be a hint that the expression was a square. This result also clearly indicates the existence of non-zero solutions to the equation $f(v, v)=0$, implying that the positive definite property is not satisfied. However, Sabet’s technical skills of operating Maple commands were not supported by algebraic-theoretical skills, and she was not able to interpret this result. Even her Maple skills appear limited; she

could have used the “factor” command in Maple, which would produce the perfect square for $f(v,v)$. She was thus neither in control over the validity of her statements, nor capable to anticipate or interpret the Maple outputs.

We summarize this student’s behavior in Table 3.

TABLE 3. ANALYSIS OF THE SOLUTION OF STUDENT 1 (SABET)

Student	Student’s theoretical behavior	Student’s non-theoretical behavior	Student’s violations of theoretical behavior	Comments
Sabet 33% of the problems she solved were T-questions	S-D : 0 S-P : 1 S-H : 0 A-L : 0 A-ML : 0	Using Maple to do calculations and to write solutions	SD : -1 SH : -1 A-L : -1 A-ML : -1	Makes inferences based on (incorrectly) memorized relationships, not on definitions. Worked with a concrete matrix A. Considered 0 a “positive” number. Did not see a contradiction between her answer to part (a) and the question in part (b)

We code Sabet’s TB as the sequence (0,1, 0, 0, 0), corresponding to the entry in column 2 of Table 3.

The Case of Student 2 (Mabi)

Mabi decided to solve four T-questions and six M-questions so we could say that she made a slightly more balanced choice than Sabet: 40% of the questions she chose could be done without Maple. She used Maple to assist her in calculations and for writing her solutions. But her technical skills in Maple were not supported by knowledge of the theory and, apart from helping her to calculate B and “see” that it is symmetric, she was not able to make use of the software for solving the second part of the problem.

One single aspect of TT in Mabi's written solution of problem M2 was the

analytic-arithmetic mode of thinking in part (a): she took an arbitrary matrix $A = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and

not a concrete numerical matrix as Sabet. She then calculated $B = AA^T$ using Maple and concluded, "It is clearly showed that the matrix B is symmetric and positive definite". She thus scored 1 on the A-L feature, but also -1 on A-ML, since she hasn't noticed the contradiction between her claim that B is positive definite and the fact that, in part (b) she was showing that f is not an inner product.

In the second part of the problem, the student produced an incorrect and nonsensical attempt of proof when claiming that f is not an inner product because $v^T B w$ is not equal to $v^T A w$. She calculated both expressions with Maple, using variables in her definitions of v and w , which, again, underscores the analytic-arithmetic aspect of her thinking. Otherwise, she scored 0 on other TT features. She didn't explicitly use the definition of the symmetric matrix, nor did she correctly use the definition of positive definite matrix, or inner product (S-D : 0). She showed no control over the validity of her reasoning and statements (S-P : 0). She appeared not to be aware of the conditional character of the statements. She stated that "an inner product is defined by $\langle v, w \rangle = v^T A w$ ", without saying anything about the conditions that A should fulfill (S-H : 0).

We summarize our analysis of Mabi's solution in Table 4.

TABLE 4. ANALYSIS OF THE SOLUTION OF STUDENT 2 (MABI)

Student	Student's theoretical behavior	Student's non-theoretical behavior	Student's violations of theoretical behavior	Comments
Mabi (40% of T-questions)	S-D : 0 S-P : 0 S-H : 0 A-L : 1 A-ML : 0	Using Maple to do calculations and to write solutions	SD : -1 A-ML : -1	Makes inferences based on observation, not on definitions. Did not see a contradiction between her answer to part (a) and the question in part (b)

The Case of Student 3 (Mabel)

Mabel attempted to solve five T-questions and five M-questions (50% of T-questions).

She used Maple to do calculations and write her solution. But the solution was very short.

Like Student 2, Mabel set the computer to calculate $B = AA^T$ with an arbitrary 3×1 matrix A and concluded that B is symmetric: "this is a symmetric matrix". For choosing an arbitrary rather than a concrete matrix, she was granted score 1 on the A-L feature. In the second part of the problem, she had Maple calculate the determinant of B . The computer returned the result 0, but the student did not write any conclusions from this output. It is possible that she knew something about the sufficient conditions for a 2×2 symmetric matrix to be positive definite (determinant > 0 and diagonal entries positive) and thought that an analogous result was true for 3×3 matrices. In this case, the 0 determinant was a sufficient argument, for her, that the given function was not an inner product. This can be seen as violation of the systemic-hypothetical feature of TT: she did not pay attention to the assumptions of the theorem about 2×2 matrices.

TABLE 5. ANALYSIS OF THE SOLUTION OF STUDENT 3 (MABEL)

Student	Student's theoretical behavior	Student's non-theoretical behavior	Student's violations of theoretical behavior	Comments
Mabel (50% of T-questions)	S-D : 0 S-P : 0 S-H : 0 A-L : 1 A-ML : 0	Using Maple to do calculations and to write solutions	SD : -1 S-H : -1	Makes inferences based on observation, not on definitions. Did not pay attention to the assumptions of a theorem and incorrectly generalized it to a higher dimension.

The Case of Student 4 (Josa)

This student's preference was clearly in favor of M-questions. She chose to solve two T-questions and eight M-questions (20% of T-questions). Still this preference cannot be associated with an effective use of Maple as a problem-solving assistant. She wasn't able to exploit Maple as help for the decisive argument.

Josa solved part (a) like Mabi and Mabel, which resulted in granting her 1 in A-L for the analytic-arithmetic mode of thinking, and 0 for the S-D feature, since she proceeded by inspection and not using a definition of symmetric matrix. She also used correct terminology and notation, confirming the score of 1 on the analytic-linguistic sensitivity feature (A-L : 1).

In part (b) she was verifying the axioms of inner product, using Maple. She used arbitrary vectors v and w , like Sabet and Mabi, but she did not define f as a function, as Sabet, but as an expression, $u := \text{Transpose}(v).B.w$. She verified the symmetry, calculating also $u2 := \text{Transpose}(w).B.v$, then $u - u2$, and asking Maple to simplify the resulting expression. The result $-0-$ was then commented upon as "the symmetry property is respected". The student then wrote a correct theoretical proof of the linearity property, scoring 1 on the systemic-proving feature (S-P : 1).

The student was not able to complete the verification of the positive definite property. Calculating $\text{Transpose}(v).B.v$ did not lead her to any clear conclusion. After “expanding” the expression, she obtained a somewhat simpler but still quite long expression. Then she defined the expression as a function f of the first component of the arbitrary vector v , and calculated $f(0)$. The output was a simpler expression:

$$v_2^2 b^2 + 2 v_2 v_3 b c + v_3^2 c^2$$

It is easier to notice that this expression is a square and conclude that, for any b and c , one can find v_2 and v_3 , not both zero, for this expression to be zero. But the student did not pursue this line of thought, perhaps because of insufficient algebraic experience and skills. We interpreted this as insufficient sensitivity to the logical structure of mathematical language and assigned 0 for analytic-metalinguistic-sensitivity feature (A-ML : 0).

TABLE 6. ANALYSIS OF THE SOLUTION OF STUDENT 4 (JOSA)

Student	Student's theoretical behavior	Student's non-theoretical behavior	Student's violations of theoretical behavior	Comments
Josa (20% of T-questions)	S-D : 0 S-P : 1 S-H : 0 A-L : 1 A-ML : 0	Using Maple to do calculations and to write solutions	S-D: -1	Makes inferences based on observation, not on definitions.

The Case of Student 5 (Brolo)

This student solved only M-questions (0% of T-questions). He used Maple successfully as a problem-solving assistant and also as a word processor. Maple output was in each case interpreted in terms of the question of the problem.

In part (a) Brolo first defined the letter variable A in Maple as a 3×1 matrix with entries a , b and c :

$$\text{"Let } A := \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{:"}$$

He preceded this definition by the word "let", which, in mathematical texts, commonly announces an introduction of a symbol and its meaning. This indicates sensitivity to the style of mathematical discourse. This, in itself, would not earn this student the score of 1 for analytic-linguistic sensitivity. But he used correct style, terminology and notation throughout his solution and, for this reason, he scored 1 on the A-L feature. He even used the Greek symbol " θ " to distinguish between the zero matrix and the zero number. The logical structure of his reasoning was also without reproach and therefore his score on A-ML was also 1.

Unlike the previously discussed students, Brolo did not stop after calculating $B = AA^T$ using Maple to just say that it can be "seen" that B is symmetric. He explicitly referred to the definition of symmetric matrix, earning score 1 for the S-D feature:

$$\text{"For } B \text{ to be symmetric, } B = B^T, \text{ or, equivalently, } B - B^T = \theta \text{."}$$

and then used Maple to calculate $B - B^T$, which produced the 3×3 zero matrix. He concluded: "So, B is symmetric."

In part (b), the student first stated a necessary condition for a matrix to define an inner product – "For f to define an inner product, the matrix B must be positive definite" – and then a necessary condition for a matrix to be positive definite: "For B to be positive definite, it must have positive eigenvalues." This explicit statement of the necessary

conditions was the reason why he scored 1 on the S-H feature. Next, he used the Maple command “*Eigenvalues(B)*” to obtain the eigenvalues 0, 0, and $a^2 + b^2 + c^2$ of B . His conclusion was:

“ Since not all eigenvalues of $B=AA^T$ are positive, the function f does not define an inner product. (Zero is nonnegative, not positive.)”

The flawless logic of this reasoning and sensitivity to the borderline distinction between “positive” and “nonnegative” resulted in score 1 on the A-ML feature.

Thus, the student’s mathematical behavior had all the five expected features of TT. His solution reveals a clear understanding and use of definitions involved, including the one of the symmetric matrix. His explanations underscore his sound understanding of the theory and reveal a certain inclination toward mathematical discourse. The student consistently engaged in proving, with an understanding of the conditional character of the propositions he was using, and had full control over terminology and notation. He correctly drew on the theoretical results to prove that the necessary conditions for f to be an inner product are not satisfied. According to our rubric, this student qualifies a theoretical thinker. We summarize his behavior in solving the problem M-2 in Table 7.

TABLE 7. ANALYSIS OF THE SOLUTION OF STUDENT 5 (BROLO)

Student	Student’s theoretical behavior	Student’s non-theoretical behavior	Student’s violations of theoretical behavior	Comments
Brolo (0% of T-questions)	S-D : 1 S-P : 1 S-H : 1 A-L : 1 A-ML : 1	Using Maple to do calculations and to write solutions		The student’s solution reads like one written by a professional mathematician.

The Case of Student 6 (Marti)

Marti was one of the two students in the whole class who preferred to tackle mostly the T-questions (80% of T-questions). He solved eight T-questions and two M-questions. This student preferred to write his exam using paper and pencil, and only used Maple for the second part of the problem M2 and for the other M-question he chose.

Marti solved part (a) of the problem using the structural definition of both the matrix B (as AA^T) and the notion of symmetric matrix. He was the only student in class who did not calculate B in order to show that it is symmetric. We can say that he was comfortable with the analytic-structural mode of thinking. By contrast, Brolo used analytic-arithmetic mode of thinking in solving part (a); even though he referred to the structural definition of symmetric matrix, he still computed the entries of the matrix $B-B^T$, instead of just using the properties of transpose, as Marti did:

$$(AA^T)^T = (A^T)^T A^T = AA^T$$

Our operationalization of the notion of theoretical thinking in terms of students' behaviors, does not distinguish between the analytic-arithmetic and the analytic-structural modes; both belong to theoretical thinking features.

Marti solved part (b) with the same technique as Brolo (he calculated the eigenvalues of the matrix B and concluded that, "Since we have eigenvalue zero, then the matrix is not positive definite and thus cannot define a real inner product." We note his mention of "real", which shows his awareness of the assumptions of the theorem (score 1 on S-H), which could be easily taken for granted, since most of the theory in the course was developed for real vector spaces. But some students were curious about how much of

this theory works also for complex numbers and asked the instructors question about that. Marti was one of them.

The summary table for Marti (Table 9) is almost the same as for Brolo, except for the percentage of chosed T-questions

TABLE 8. ANALYSIS OF THE SOLUTION OF STUDENT 6 (MARTI)

Student	Student's theoretical behavior	Student's non-theoretical behavior	Student's violations of theoretical behavior	Comments
Marti (80% of T-questions)	S-D : 1 S-P : 1 S-H : 1 A-L : 1 A-ML : 1	Using Maple to do calculations and to write solutions		The only student to have used the analytic-structural mode of thinking in solving part (a)

The Case of Student 7 (Rami)

Rami privileged the M-questions; he tackled three T-questions and seven M-questions (30% of T-questions). Like most of the other students discussed above (except Marti) he wrote his solution entirely in the Maple file.

In part (a) Rami defined $A := \langle\langle a, b, c \rangle\rangle$ in Maple, i.e. as a one-column matrix, and not as a vector, which shows a non-trivial (linguistic and meta-linguistic) sensitivity to the subtle difference between the two objects (whence scores 1 for A-L and A-ML). Then he first computed $B = AA^T$ and then *Transpose(B)* in Maple, concluding: "Therefore, since $B^T = B$, then it is symmetric". This is an action of proving (S-P : 1), of course, but part of the argument is visual evidence, and therefore, we see this behavior as a violation of the systemic-definitional feature (S-D : -1).

In part (b), the student computed the eigenvalues of B using Maple and concluded, "Since the eigenvalues of the matrix are not all positive, then it is not an inner product." The link of the condition on eigenvalues with inner product is not mentioned

(as it was in the solutions of Marti and Brolo) and therefore his solution shows no evidence of hypothetical thinking (S-H : 0).

Table 9 summarizes this analysis.

TABLE 9. ANALYSIS OF THE SOLUTION OF STUDENT 7 (RAMI)

Student	Student's theoretical behavior	Student's non-theoretical behavior	Student's violations of theoretical behavior	Comments
Rami (30% of T-questions)	S-D : 0 S-P : 1 S-H : 0 A-L : 1 A-ML : 1	Using Maple to do calculations and to write solutions	S-D: -1	The only student to distinguish between a vector and a one-column matrix.

The case of Student 8 (Sanna)

This student chose to solve six T-questions by paper and pencil and the other four M-questions on the computer (60% of T-questions). None of the five expected TT features have been detected when analyzing her written solution. She didn't use any of the definitions involved (S-D : 0), and she used a particular matrix instead of an arbitrary 3×1 matrix (we considered this a violation of the systemic-hypothetical feature, S-H : -1), together with incorrect terminology, and several nonsensical statements. There were many typographical errors. This explains her score of -1 on the A-L feature. The student also violated the analytic-metalinguistic feature of TT (A-ML: -1) when, in part (b) she confused the implication *If p then q* (in Th. 7.8) with *If not p then not q* ("if B does not satisfy the positive definite, then the function $v^T B w$ cannot be an inner product").

Her answer to part (a) contained a concrete matrix $A := \langle 1, -1, 5 \rangle$, a calculation of the matrix $B = AA^T$ and the following verbal explanation:

"B is symmetric because a 3×1 matrix was multiplies with a 1×3 matrix giving a 3×3 matrix with entries which are squares of entries of A. The row space and column space of B is the column space of A which are which are

multiplied by the entries themselves. Therefore it gives a symmetric matrix."

We have reproduced all the typographical mistakes of the student. We note the absence of reference to the definition of symmetric matrix, the erroneous use of terms "column space" and "row space", and the surprising observation that the entries of B are squares of entries of A (in spite of having obtained negative entries in B). It may well be that the word "squares" is also used in a meaning which is different from the conventional one in this context.

The student is not in control of the validity of her statements, nor does she ask herself whether the statement is still valid for any 3×1 matrix A .

In part (b) Sanna tried different arguments, but she appeared not to know which was correct and why (S-P: 0). She calculated $v^T B w$ for arbitrary v and w (letter components) and her particular matrix B , expanded the complicated expression she obtained and then abandoned this path. She then calculated the determinant of B , obtaining 0, and concluding (like Mabel), that the function is not an inner product (in fact she says, "not an inner product *space*" – note the erroneous terminology) since "for $f(v, w)$ to be an inner product, $\det(B) > 0$ ", extrapolating from the 2-dimensional case, and again violating the systemic-hypothetical feature. She did not stop here, however, and continued her solution, calculating the eigenvalues of B , obtaining two zero eigenvalues and concluding:

"The eigenvalues of B are not all positive therefore it does not satisfy the positive definite property. According to Theorem 7.8 p. 507, if B does not satisfy the positive definite, the the function $v^T B w$ cannot be an inner product."

We see here the violation of the logical law already mentioned above.

We summarize this analysis in Table 10 below.

TABLE 10. ANALYSIS OF THE SOLUTION OF STUDENT 8 (SANNA)

Student	Student's theoretical behavior	Student's non-theoretical behavior	Student's violations of theoretical behavior	Comments
Sanna (60% of T-questions)	S-D : 0 S-P : 0 S-H : 0 A-L : 0 A-ML : 0	Using Maple to do calculations and to write solutions	S-D : -1 S-H : -1 A-L : -1 A-ML : -1	The student ignores definitions. Uses a concrete matrix A. Uses erroneous terminology. Makes logical mistakes.

Summary of students' theoretical behaviors and violations of theoretical behavior

We summarize our analyses of students' behaviors in solving the question M-2 in Table 11.

The most violated feature of theoretical behavior is the systemic definitional feature. Six out of 8 students made inferences based on observation or incorrectly memorized relationships and not on definitions. Three out of 8 students violated the systemic hypothetical feature by ignoring the assumptions of the statement or working in a concrete case. Another feature that has been violated several times (3 out of 8) is analytic-meta linguistic sensitivity (logical mistakes or lack of sensitivity to contradictions).

On the other hand, the most frequent features of theoretical behavior manifested by students have been systemic-proving (5 out of 8) and analytic linguistic sensitivity (6 out of 8)

TABLE 11. SUMMARY OF STUDENTS' THEORETICAL BEHAVIOR

Student	TT features				
	S-D	S-P	S-H	A-L	A-ML
Sabet	-1	1	-1	-1	-1
Mabi	-1	0	0	1	-1
Mabel	-1	0	-1	1	0
Josa	-1	1	0	1	0
Brolo	1	1	1	1	1
Marti	1	1	1	1	1
Rami	-1	1	0	1	1
Sanna	-1	0	-1	-1	-1
fTB	2 out of 8	5 out of 8	2 out of 8	6 out of 8	2 out of 8
fvTB	6 out of 8	0	3 out of 8	2 out of 8	3 out of 8

Legend: fTB = frequency of theoretical behavior;
fvTB = frequency of violation of theoretical behavior.

Chapter7: Conclusions and Discussion

In this last chapter we will discuss the research findings by formulating our answers to the research questions. We will also try to place our study within the context of the research area of interest. Suggestions for future research and the implications of the findings for teaching will be presented.

7.1 Research Context

Research acknowledges that, in the presence of technology, students face new challenges since they have to understand and adapt the software commands to a specific mathematical task (Artigue, 2001, 2005). Students are now able to work with complex expressions; they have to understand how a certain instrumented technique relates to their prior knowledge acquired in the traditional environment. They also have to understand how the machine operates; they have to make decisions about the usefulness of the outputs returned by the machine, as well as anticipate the possible results and learn when it is worthwhile to use the utilities afforded by the instrument (Lagrange, 2005). At times, mathematical software adds a degree of complexity that might be perceived as a burden, either for the weaker students or even for those who have already master the apper-and-pencil techniques. Controlling the machine is not an easy task. There are electronic, syntax and organization constraints that have to be managed by the users (Trouche, 2005). It is now the case that mathematical reasoning is tightly connected to an effective instrumentation of the tool (Ruthven, 2002).

Literature abounds with studies about the potential role of computer algebra systems (CAS) in teaching and learning mathematics. It is believed that using CAS the introduction of new concepts can be easily motivated (Artigue, 2005). A CAS may be used as a problem solving assistant, or as a tool for visualization and interpretation of mathematical results, but also as an environment for exploration, discovery and pattern recognition (Lagrange, 2005; Thomas and Hong, 2004; Berry et al., 1994). Other researchers (Drijvers and Gravemeijer, 2005) suggest that the CAS environment promotes understanding of the steps involved in the solution process. CAS is also claimed to be a facilitator of conversion and coordination between the algebraic and graphical registers (Winsløw, 2003). On the other hand, there is the risk, especially for the weaker students, that understanding and reasoning will be replaced by push-button techniques and that they will rely too much on CAS to assist in problem solving (Thomas & Hong, 2004; Crowe and Zand, 2000).

At this point of time, the use of CAS in the teaching and learning mathematics has not been fully integrated with other means of teaching and learning mathematics. In spite of a large body of research, the use of CAS has not been institutionalized. There are research experiments (Trouche, 2005; Drijvers and Gravemeijer, 2005) that show the potential of CAS to bring about conceptual gains in the mathematics class, but these experiments have not been widely reproduced or applied (Artigue, 2005). Integration of CAS in the mathematics classroom is not just a matter of adding another tool. The entire practice has to be modified to accommodate the specific needs of the instrumented work. A new theoretical discourse has to be created to justify the new instrumented- assisted tasks and techniques.

Researchers acknowledge the usefulness of technology for teaching and learning linear algebra (LACSG, 1997). CAS (Maple) has been used with various pedagogical purposes: to exploit the effect of surprise to motivate further investigations aiming to discover important properties of matrices; to generate discussion on important theoretical results for clarifying concepts and reveal misconceptions; to introduce new and more advanced concepts before discussing them in class through Maple lab worksheets (Hillel, 2001). Several mathematics professors who are engaged in teaching linear algebra at the university level propose ways of integrating CAS (MATLAB, MAPLE, MATHEMATICA) in the linear algebra courses (Day, 1997). In their technology-assisted classes, CAS released students from the burden of tedious computations and favored the focus on concepts by allowing students to investigate a large number of examples. Technology in teaching linear algebra helps with the visualization in 3D of randomly generated vectors and their images under a transformation, but it is also an experimental environment in which students can “play” with vectors and matrices using various representations.

On the other hand, it has been also argued that a more concrete approach to teach linear algebra by using visualization (in Cabri-geometry dynamic environment) and examples in low dimensions may lead to irrelevant interpretations and misunderstandings (Sierpinska et al., 1999; Sierpinska 2000).

Our study is somehow different from other studies (Sierpinska et al. 2002; Bobos 2004) in that it focuses on exploring the theoretical thinking of students in a computer-assisted environment. Previous research explored theoretical thinking in linear algebra in the context of the traditional environment. Theoretical thinking was examined as a

possible factor for high achievement, and researchers found that this is a necessary but not a sufficient factor for a good understanding of linear algebra. The research design and the interpretation of data are founded on the model of theoretical thinking, which Sierpinska et al. developed while taking into account the existing body of literature on that subject. The theory and the postulated definitions were created for methodological purposes as means for analyzing and interpreting students' thinking. We used this model and its categories to analyze our data.

7.2 Research findings

We now address the research questions by formulating some answers together with the implications of these findings for teaching and further research.

How do students think in a computer-assisted linear algebra course?

Overall, we cannot say that the group of students whose solutions we studied is made of theoretical thinkers. Only two out of the eight students provided a solution that demonstrated thinking having all the four theoretical thinking features necessary to solve the problem. By contrast, five out of the eight students showed a poor theoretical behavior or did not show any theoretical behavior at all.

The two students, who scored 1 in each of the five TT features necessary to solve the problem, used Maple efficiently as a computational assistant, and succeeded also in showing their understanding of the concepts involved and of the relations among them. Since they were in command of the solution process, they have put the software to good

use, and they knew just to what extent it is worthwhile using it. We could say that the control comes from the understanding and use of the theory. Because the task was meant to assess their knowledge and understanding of the theory, it happened that the high score in TT coincided with the maximum grade for the problem (10 points). We cannot venture to say that these two students achieved the highest mark specifically because they were theoretically inclined. In the study undertaken by Sierpinska et al. (2002), TT was shown not to be a sufficient condition for high achievement in linear algebra courses.

On the other hand, the five students who scored low in TT violated at least one of TT features. They all involved Maple in their solution, but have not managed to control the software so as to correctly solve the task. One student (Sabet) used Maple to verify the inner product axioms, but still didn't know how to handle the final argument, probably because she lacked the algebraic and conceptual knowledge necessary to follow this tedious path. This phenomenon of over-reliance on CAS, which in our case prevented students from trying other more economical strategies, has been already mentioned in the literature (Thomas and Hong, 2004; Crowe and Zand, 2000). Other students used Maple to calculate $v^T B w$ for any v and w , but they either produced nonsensical inferences about the computer results (see Mabi's work in the Appendix), or suspended this strategy and tried another one (see Sanna's work in the Appendix). This points to a possible incapacity to evaluate the relevance of a technique or to anticipate computer outputs. The "oscillation" phenomenon of students trying several techniques and strategies without being in control of neither one has already been observed (Defouad, 2000). These students' method of work was close to what Trouche (2005) calls "automatic method": cut and paste strategies from memorized solutions, trial and error

procedures without methods of validating the outputs of the machine. The above-mentioned phenomenon has been initially discovered in a context of working with symbolic calculators, but, as we have showed before, there are similar situations when working with Maple as well. Our conjecture is that such phenomena may occur while working with any CAS.

We have to keep in mind that the written samples, which we analyzed, have been given in a final exam frame. Due to the time limit, students had to make pragmatic decisions, which might have limited at some point the development of their mathematical discourse.

In the presence of CAS, what features of theoretical thinking do students mobilize?

It seems that the most frequently manifested features were those associated with the proving engagement and linguistic sensitivity. This is probably due to the nature of the subject – students expect to be asked to produce a proof using adequate terminology in a mathematics course. Many students at this level have a sense of what it means to prove a statement. Still, the engagement with proving is not complete, since many students do not have the habit to validate their solutions. The question “How do I know that my solution is correct?” is not part of the solution process. The other features of TT (systemic-definitional, systemic- hypothetical, and analytic-meta-linguistic-sensitivity) were the least frequent.

The most frequently violated TT feature (6 of 8 students) concerns the systemic-definitional character of TT. Students based their conclusions on inspection and observation of the matrix B , rather than on the definition of the symmetric matrix.

Our study is taking into account a small sample of students; consequently, we cannot generalize these observations and say that the systemic-definitional feature of TT is more likely to be violated than other features. Further research on the frequency of these TT features in a Maple-assisted environment is needed. Bobos (2004), who studied students' TT in a paper and pencil environment, suggests that TT behavior may be dependent on the nature of the mathematical task.

What kind of obstacles to the understanding of linear algebra students encounter? Which of them are specific to CAS environment?

The study revealed that some students have a statistical view of the idea of generality: for example, a randomly generated matrix stands as a substitute for generality. This kind of violation of TT (systemic-hypothetical thinking) is specific to CAS-assisted environment in which students very often use the “RandomMatrix” command. We could say that this statistical way of thinking, which was useful in other contexts, acts as an epistemological obstacle that prevents students from grasping the nature of mathematical generality.

There is the risk for students to think in terms of commands (for example, the norm could be seen as a Maple command rather than an axiomatically defined function). Similarly, the literature mentions the case of “automatic transportation (Trouche, 2005) that is students input all the data and then look for a command capable of giving the

solution right away. Teaching with the computer might facilitate thinking in terms of commands and therefore create a didactical obstacle. In our case, the course instructor was very much aware of this, and tried to turn this obstacle into an opportunity for discussion and learning.

We also found that sometimes the instrument acts as an obstacle to successfully accomplishing a task. The student is not capable to decide when the CAS is useful and when it is not. There is the risk of relying too much on Maple calculations with the consequence of choosing a solution path that is not economical and which leads to complex situations that students cannot manage (like our students who began to check the axioms of the inner product, but did not succeed in managing the positive definite argument which required algebraic skills and a good sense of a suitable command). Besides phenomena like “oscillation” and “automatic transportation”, other phenomena linked to the organization of students’ instrumented work were cited in the literature: “localized determination” (repeating the same irrelevant technique although it is proving to be ineffective, without changing registers), “over-checking” (multiple checks) or “zapping” (changing windows without allowing enough time for analyzing each output) (Defouad, 2002; Trouche 2005).

It is not obvious that techniques contribute to students’ understanding (and thus have an epistemic value). More often than not, the discourse that should go together with the technique and show students’ understanding of the concept is not there. Communicating mathematical thinking is not the main concern. It is more important to give a final answer rather than explain the process that leads to it. Students remain at most at the level of techniques, their written solutions are very abbreviated and the reader

has to guess what is the reasoning behind. It seems that students think that computer outputs speak for themselves. This pragmatic behavior could be seen as an obstacle to knowledge development.

Teaching and learning linear algebra changes in the presence of technology. When CAS like Maple or Matlab are used only as computational devices, the way students think and their means of reaching conceptual understanding is not different from the traditional environment. The computer alleviates the burden of calculations and saves precious time, which may in turn be allotted to thinking and discussion of important concepts and how they are related to each other. In this scenario, the technology is very useful but still dispensable. Besides the role of a calculator, there is the opportunity to use CAS for pedagogical purposes. There might be situations when students understand and apply concepts easier if they are asked to explore and experiment certain concepts with CAS. For example, prompted by their instructor, students may discover by themselves the properties of matrices (e.g. the transpose) by working with multiple examples in Maple or Matlab, or they can visualize the eigenvector / eigenvalue concepts and link easily the geometrical and algebraic representations. Other properties of matrices may be explored; the computer could help in generating counterexamples to check for example that $A^{-1} + B^{-1} \neq (A + B)^{-1}$ etc. The teacher has to guide and plan in advance these activities so as to leave a degree of flexibility and freedom to students, at the same time keeping the pace and didactical objectives imposed by the curriculum. In this enterprise, the teacher has to be supported by the institution, since a lot of time and effort will be needed to accommodate and integrate these new pioneering didactical strategies.

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Appendix: Students' Written Solutions

Case 1 (Sabet)

▼ Student's solution

a)

$A := \text{RandomMatrix}(3, 1, \text{generator} = 1 \dots 4);$

$$\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$$

(14.1.1)

$B := A.\text{Transpose}(A);$

$$\begin{bmatrix} 9 & 6 & 6 \\ 6 & 4 & 4 \\ 6 & 4 & 4 \end{bmatrix}$$

(14.1.2)

$\text{Eigenvalues}(B);$

$$\begin{bmatrix} 0 \\ 0 \\ 17 \end{bmatrix}$$

(14.1.3)

B is symmetric because the eigenvalues are all positive

b)

$v := \langle v1, v2, v3 \rangle :$

$w := \langle w1, w2, w3 \rangle :$

$f := (v, w) \rightarrow \text{Transpose}(v).B.w;$

$(v, w) \rightarrow \text{Typesetting:-delayDotProduct}$

(14.1.4)

$(\text{Typesetting:-delayDotProduct}(\text{LinearAlgebra:-Transpose}(v), B), w)$

$f1 := f(v, w);$

$w1 (9 v1 + 6 v2 + 6 v3) + w2 (6 v1 + 4 v2 + 4 v3) + w3 (6 v1 + 4 v2 + 4 v3)$ (14.1.5)

$f2 := f(w, v);$

$v1 (9 w1 + 6 w2 + 6 w3) + v2 (6 w1 + 4 w2 + 4 w3) + v3 (6 w1 + 4 w2 + 4 w3)$ (14.1.6)

$f1 - f2;$

$w1 (9 v1 + 6 v2 + 6 v3) + w2 (6 v1 + 4 v2 + 4 v3) + w3 (6 v1 + 4 v2$ (14.1.7)

$+ 4 v3) - v1 (9 w1 + 6 w2 + 6 w3) - v2 (6 w1 + 4 w2 + 4 w3) - v3 (6 w1$
 $+ 4 w2 + 4 w3)$

$\text{simplify}(\%);$

0

(14.1.8)

$f(v, v);$

$v1 (9 v1 + 6 v2 + 6 v3) + v2 (6 v1 + 4 v2 + 4 v3) + v3 (6 v1 + 4 v2 + 4 v3)$ (14.1.9)

$\text{eq1} := \text{simplify}(\%);$

$9 v1^2 + 12 v1 v2 + 12 v1 v3 + 4 v2^2 + 8 v2 v3 + 4 v3^2$ (14.1.10)

$\text{solve}(\{\text{eq1}\}, \{v1, v2, v3\});$

$\left\{ v1 = -\frac{2}{3} v2 - \frac{2}{3} v3, v2 = v2, v3 = v3 \right\}, \left\{ v1 = -\frac{2}{3} v2 - \frac{2}{3} v3, v2 = v2, v3 = v3 \right\}$ (14.1.11)

Case 2 (Mabi)

▼ Student's solution

a

$$A := \langle x, y, z \rangle; 1$$

a

(14.1.1)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(14.1.2)

$$B := \text{Typesetting:-delayDotProduct}(A, \text{Transpose}(A)); 1$$

$$\begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}$$

(14.1.3)

It is clearly showed that the matrix B is symmetric and positive definite

An inner product is defined by $\langle v, w \rangle = v^T A w$

(14.1.4)

But lets prove that b :

$f(v, w) = v^T A w \neq f(v, w)$ and $f(v, w) = v^T B w$

(14.1.5)

$$v := \langle x1, x2, x3 \rangle; 1$$

$$\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix}$$

(14.1.6)

$$w := \langle y1, y2, y3 \rangle; 1$$

$$\begin{bmatrix} y1 \\ y2 \\ y3 \end{bmatrix}$$

(14.1.7)

$$C := \text{Typesetting:-delayDotProduct}(\text{Typesetting:-delayDotProduct}(\text{Transpose}(v), A), w); 1$$

$$(x x1 + y x2 + z x3) \cdot \begin{bmatrix} y1 \\ y2 \\ y3 \end{bmatrix}$$

(14.1.8)

$$F := \text{Typesetting:-delayDotProduct}(\text{Typesetting:-delayDotProduct}(\text{Transpose}(v), B), w); 1$$

$$y1 (x1 x^2 + x2 x y + x3 x z) + y2 (x1 x y + x2 y^2 + x3 y z) + y3 (x1 x z + x2 y z + x3 z^2) \quad (14.1.9)$$

$C \neq F$

Since C defines an inner product and F doesn't equal it, therefore $(v^T B w)$ is not an inner product.

Case 3 (Mabel)

▼ Student's solution

(a)

$$A := \begin{bmatrix} x \\ y \\ z \end{bmatrix};$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(14.1.1)

$$B := \text{Typesetting:-delayDotProduct}(A, \text{Transpose}(A)); 1$$

$$\begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}$$

(14.1.2)

this is a symmetric matrix

b)

$$\text{Determinant}(B); 1$$

0

(14.1.3)

Case 4 (Josa)

▼ Student's solution

with(LinearAlgebra) :

A := < a, b, c >;

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad (14.1.1)$$

B := A.Transpose(A);

$$\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix} \quad (14.1.2)$$

We can see that be is a symmetric matrix

v := < v1, v2, v3 >; w := < w1, w2, w3 >;

u := Transpose(v).B.w;

$$w1 (v1 a^2 + v2 ab + v3 ac) + w2 (v1 ab + v2 b^2 + v3 bc) + w3 (v1 ac + v2 bc + v3 c^2) \quad (14.1.3)$$

u2 := Transpose(w).B.v

$$v1 (w1 a^2 + w2 ab + w3 ac) + v2 (w1 ab + w2 b^2 + w3 bc) + v3 (w1 ac + w2 bc + w3 c^2) \quad (14.1.4)$$

u-u2;

$$w1 (v1 a^2 + v2 ab + v3 ac) + w2 (v1 ab + v2 b^2 + v3 bc) + w3 (v1 ac + v2 bc + v3 c^2) - v1 (w1 a^2 + w2 ab + w3 ac) - v2 (w1 ab + w2 b^2 + w3 bc) - v3 (w1 ac + w2 bc + w3 c^2) \quad (14.1.5)$$

simplify(%);

0

(14.1.6)

the symmetry property is respected

$$\langle av1 + bv2, w \rangle = (av1 + bv2)^T Bw = av1^T Bw + bv2^T Bw = a \langle v1, w \rangle + b \langle v2, w \rangle$$

linearity property is respected

Transpose(v).B.v

$$v1 (v1 a^2 + v2 ab + v3 ac) + v2 (v1 ab + v2 b^2 + v3 bc) + v3 (v1 ac + v2 bc + v3 c^2) \quad (14.1.7)$$

expand(%);

$$v1^2 a^2 + 2 v1 v2 ab + 2 v1 v3 ac + v2^2 b^2 + 2 v2 v3 bc + v3^2 c^2 \quad (14.1.8)$$

$$f := v1 \rightarrow v1^2 a^2 + 2 v1 v2 ab + 2 v1 v3 ac + v2^2 b^2 + 2 v2 v3 bc + v3^2 c^2 \quad (14.1.9)$$

f(0);

$$v2^2 b^2 + 2 v2 v3 bc + v3^2 c^2 \quad (14.1.10)$$

Case 5 (Brolo)

▼ Student's solution

(a)

Let

$$A := \begin{bmatrix} a \\ b \\ c \end{bmatrix} :$$

$$B := \text{Typesetting:-delayDotProduct}(A, \text{Transpose}(A))$$

$$\begin{bmatrix} a^2 & a b & a c \\ a b & b^2 & b c \\ a c & b c & c^2 \end{bmatrix} \quad (14.1.1)$$

For B to be symmetric, $B = B^T$, or equivalently, $B - B^T = \mathbf{0}$.

$$B - \text{Transpose}(B)$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (14.1.2)$$

So, B is symmetric.

(b)

For f to define an inner product, the matrix B must be positive definite. For B to be positive definite, it must have positive eigenvalues.

$$\text{Eigenvalues}(B)$$

$$\begin{bmatrix} 0 \\ 0 \\ a^2 + b^2 + c^2 \end{bmatrix} \quad (14.1.3)$$

Since not all eigenvalues of $B = AA^T$ are positive, the function f does not define an inner product. (Zero is nonnegative, but not positive.)

Case 6 (Marti)

M₂

$$(a) (AA^T)^T = (A^T)^T A^T = AA^T$$

hence B is symmetric.

(b) View Maple (We get eigenvalue zero).

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with(LinearAlgebra); -1
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> M1
```

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>
```

```
> C := <<c11, c21, c31>>; 1
```

$$C := \begin{bmatrix} c11 \\ c21 \\ c31 \end{bmatrix}$$

(1)

```
> F := Typesetting:-delayDotProduct(C, Transpose(C)); 1
```

$$F := \begin{bmatrix} c11^2 & c11 c21 & c11 c31 \\ c11 c21 & c21^2 & c21 c31 \\ c11 c31 & c21 c31 & c31^2 \end{bmatrix}$$

(2)

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> Eigenvalues(F); 1
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$$\begin{bmatrix} 0 \\ 0 \\ c11^2 + c21^2 + c31^2 \end{bmatrix}$$

(3)

Since we have eigenvalue zero, then the matrix is not positive definite and thus cannot define a real inner product. (Theorem 8.37)

Case 7 (Rami)

▼ Student's solution

with (LinearAlgebra); -1

A := <<a, b, c>>; 1

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

(14.1.1)

B := Typesetting:-delayDotProduct(A, Transpose(A)); 1

$$\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$$

(14.1.2)

Transpose(B); 1

$$\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$$

(14.1.3)

therefore since $B^T = B$ then it is symmetric

Eigenvalues(B); 1

(14.1.4)

$$\begin{bmatrix} 0 \\ 0 \\ c^2 + b^2 + a^2 \end{bmatrix}$$

(14.1.4)

Since the eigenvalues of the matrix are not all positive then it is not an inner product

Case 8 (Sanna)

▼ Student's solution

$$A := \langle 1, -1, 5 \rangle; 1$$

$$\begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} \quad (14.1.1)$$

$$B := \text{Typesetting:-delayDotProduct}(A, \text{Transpose}(A)); 1$$

$$\begin{bmatrix} 1 & -1 & 5 \\ -1 & 1 & -5 \\ 5 & -5 & 25 \end{bmatrix} \quad (14.1.2)$$

B is symmetric because a 3 x 1 matrix was multiplies with a 1 x 3 matrix giving a 3 x 3 matrix with entries which are the squares of entries of A. The row space and column space of B is the column space of A which are which are multiplied by the entries themselves. Therefore it gives a symmetric matrix.

$$v := \langle x_1, x_2, x_3 \rangle; 1$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (14.1.3)$$

$$w := \langle y_1, y_2, y_3 \rangle; 1$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad (14.1.4)$$

$$\text{Typesetting:-delayDotProduct}(\text{Typesetting:-delayDotProduct}(\text{Transpose}(v), B), w); 1$$

$$y_1 (x_1 - x_2 + 5 x_3) + y_2 (-x_1 + x_2 - 5 x_3) + y_3 (5 x_1 - 5 x_2 + 25 x_3) \quad (14.1.5)$$

$$\text{expand}(\%); 1$$

$$y_1 x_1 - y_1 x_2 + 5 y_1 x_3 - y_2 x_1 + y_2 x_2 - 5 y_2 x_3 + 5 y_3 x_1 - 5 y_3 x_2 + 25 y_3 x_3 \quad (14.1.6)$$

$$\det(B); 1$$

$$0 \quad (14.1.7)$$

for $f(v, w)$ to be an inner product function, $\det(B) > 0$,
in this case it is equal to 0. Therefore $f(v, w)$ is not an inner product space.

$$\text{Eigenvalues}(B); 1$$

$$\begin{bmatrix} 0 \\ 0 \\ 27 \end{bmatrix} \quad (14.1.8)$$

The eigenvalues of B are not all positive
therefore it does not satisfy the positive definite property. According to theorem 7.8 p 507,
if B does not satisfy the positive definite, the function $v^T B w$ cannot be an inner product.